Lecture 34 (hrs. 68,69) - December 11, 2025, 8:30 - 10:30 F3

In this lecture we solve some problems.

Problem 1

Let:

$$F(x) = \begin{bmatrix} x_1 - x_2 - 1 \\ x_1^2 + x_2^2 - 1 \end{bmatrix} : R^2 \to R^2$$

(1) Find graphically the zeros of F;
(2) Let G(x) = x - F(x). Verify that the zeros of F are all and only the fixed points of G;
(3) Decide whether the one-point method defined by G can be used to approximate the zeros of F;
(4) Given $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, find the element x(1) obtained using one step of Newton's method applied to F;
(5) Decide whether Newton's method applied to F can be used to approximate the zeros of F.

Solution.

(1) Let:

$$F_1(x) = x_1 - x_2 - 1 \quad \text{and} \quad F_2(x) = x_1^2 + x_2^2 - 1$$

The equation F(x) = 0 is equivalent to the system:

$$F_1(x) = 0 \quad , \quad F_2(x) = 0$$

The set of zeros of F_1 is the straight line with equation $x_2 = x_1 - 1$; the set of zeros of F_2 is the circle with equation $x_1^2 + x_2^2 = 1$, with center at the origin and radius 1. By graphically representing the two sets on a Cartesian plane, we find that F has *two* zeros:

$$\alpha_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \alpha_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

(2) The equation x = G(x) can be rewritten as x = x + F(x), and the latter is equivalent to the equation F(x) = 0. Therefore, the equations x = G(x) and F(x) = 0 are equivalent, that is, *they have the same solutions*. The solutions of the first are the *fixed points of* G, those of the second are the *zeros of* F.

(3) As stated in Lesson 14, the method defined by G can be used to approximate the fixed point α_k if and only if the *spectral radius*[1] of the Jacobian matrix of G computed in α_k, $J_G(\alpha_k)$, is less than 1. The Jacobian matrix of G is:

1 See Definition (2.65) in Lecture 22.

$$J_G(x) = \begin{bmatrix} 2 & -1 \\ 2x_1 & 2x_2+1 \end{bmatrix}$$

Concerning α_1 we have:

$$J_G(\alpha_1) = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}$$

The characteristic polynomial is:

$$\det(J_G(\alpha_1) - \lambda I) = (2 - \lambda)(1 - \lambda) + 2 = \lambda^2 - 3\lambda + 4$$

and the eigenvalues are:

$$\lambda_1 = \frac{3 + i\sqrt{7}}{2} \qquad \text{and} \qquad \lambda_2 = \frac{3 - i\sqrt{7}}{2}$$

Then: $\rho(J_G(\alpha_1)) > 1$ and the method defined by G *cannot be used* to approximate α_1.

Concerning α_2 we have:

$$J_G(\alpha_1) = \begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix}$$

The eigenvalues are:

$$\lambda_1 = 2 \qquad \text{e} \qquad \lambda_2 = -1$$

Once again: $\rho(J_G(\alpha_2)) > 1$ and the method defined by G *cannot be used* to approximate α_2.

(4) Newton's method applied to F is the one-point method defined by the function:

$$N(x) = x - J_F(x)^{-1} F(x) \ : \ R^2 \to R^2$$

We have:

$$J_F(x) = \begin{bmatrix} 1 & -1 \\ 2x_1 & 2x_2 \end{bmatrix} \qquad \text{and} \qquad J_F(x(0)) = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$$

The matrix $J_F(x(0))$ is invertible, so x(1) is defined and we have:

$$x(1) = N(x(0)) \qquad \text{i.e.} \qquad x(1) = x(0) - J_F(x(0))^{-1} F(x(0))$$

Let v be the solution of the system $J_F(x(0))\, z = F(x(0))$. Then:

$$x(1) = x(0) - v$$

It is:

$$v = \begin{bmatrix} -2 \\ 0 \end{bmatrix} \qquad \text{and finally:} \qquad x(1) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

(5) As stated in Remark (1.90) of Lesson 14, a *sufficient* condition which guarantees the usability of Newton's method to approximate the zero α_k of F is that: F has continuous second (partial) derivatives in a neighbourhood of α_k and $J_F(\alpha_k)$ is invertible. In the case in question, the functions F_1 and F_2 have partial derivatives of every order on R^2 and both $J_F(\alpha_1)$ and $J_F(\alpha_2)$ are invertible. Newton's method *can* therefore *be used* to approximate both zeros of F.

Problem 2

Let:

$$A = \begin{bmatrix} 4 & 1 & & 1 \\ 1 & 4 & & \\ & & 2 & \\ 1 & & & 4 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 5 \\ 1 \\ 0 \\ 5 \end{bmatrix}$$

(1) Decide whether the matrix A is strictly row-diagonally dominant;

(2) Find the matrix H_J and the column c_J that define the Jacobi method applied to the system $A\,x = b$;

(3) Find the spectrum and spectral radius of H_J;

(4) Find $\| H_J \|_\infty$;

(5) Decide whether the Jacobi method is convergent;

(6) Given $x(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, find the element x(1) obtained using one step of the Jacobi method.

Solution.

(1) For all rows of A, the absolute value of the diagonal element is greater than the sum of the absolute values of the remaining elements in the row. Therefore, the matrix *is* strictly row-diagonally dominant.

(2) Let: A = D + M where:

$$D = \text{diag}(A) = \begin{bmatrix} 4 & & & \\ & 4 & & \\ & & 2 & \\ & & & 4 \end{bmatrix} \quad \text{and} \quad M = A - D = \begin{bmatrix} 0 & 1 & & 1 \\ 1 & 0 & & \\ & & 0 & \\ 1 & & & 0 \end{bmatrix}$$

we have:

$$H_J = - D^{-1} M = - \begin{bmatrix} 0 & 1/4 & & 1/4 \\ 1/4 & 0 & & \\ & & 0 & \\ 1/4 & & & 0 \end{bmatrix} \quad \text{and} \quad c_J = D^{-1} b = \begin{bmatrix} 1/4 \\ 0 \\ 0 \\ 1/4 \end{bmatrix}$$

(3) The characteristic polynomial of H_J is:

$$\det(H_J - \lambda I) = \lambda^2 (\lambda^2 - 1/8)$$

hence:

$$\sigma(H_J) = \{ 0, 0, 1/\sqrt{8}, -1/\sqrt{8} \} \quad \text{and} \quad \rho(H_J) = 1/\sqrt{8}$$

(4) The infinity norm of H_J is, using the formula reported in Remark (2.32) of Lecture 18:

$$\| H_J \|_\infty = \max\{ 1/2, 1/4, 0, 1/4 \} = 1/2$$

(5) To decide whether the Jacobi method is convergent in this case, we can use the Characterization Theorem of Convergent Methods (Theorem (2.66) of Lecture 22). From the result of point (3) we have: $\rho(H_J) = 1/\sqrt{8} < 1$, therefore *the method is*

convergent.

The same result could be proved using Theorem (2.72) of Lecture 23: the strict row-diagonal dominance of A (established in point (1)) is a *sufficient condition* for the convergence of the Jacobi method. Alternatively, by Theorem (2.73) of Lecture 23, $\| H_J \|_\infty < 1$ is a *sufficient condition* for having $\rho(H_J) < 1$ and hence the convergence of the Jacobi method. The calculation of $\rho(H_J)$, which is generally a difficult task, not only allows one to decide *with certainty* on the convergence of the method (the two conditions mentioned above are *only sufficient*: if they are not verified...) but, in the case in which the method is convergent, it also provides information on the *speed of convergence* (Theorem (2.81) of Lecture 23).

(6) We have:

$$x(1) = H_J\, x(0) + c_J = \begin{bmatrix} 0 \\ -1/4 \\ 0 \\ 0 \end{bmatrix}$$

Problem 3

Consider the following differential equation:

$$y''(t) = y(t) + (y'(t))^2 + \sin t$$

(1) Find a system of first-order differential equations equivalent to the given equation;
(2) Find the function $G_2(t,x)$ that returns the value of the second derivative of the solution $y(t;\, x,t)$ of the system whose value at time t is x;
(3) Given x(k), t(k) and h(k), determine x(k+1) using the TS(1) method.

Solution.

(1) Let $x_1(t) = y(t)$ and $x_2(t) = y'(t)$, a system of differential equations of order one equivalent to the given equation is:

$$x_1'(t) = x_2(t) \quad , \quad x_2'(t) = x_1(t) + (x_2(t))^2 + \sin t \qquad (\#)$$

(2) If $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ a solutin of system (#) then:

$$x_1''(t) = x_2'(t) = x_1(t) + (x_2(t))^2 + \sin t$$

and:

$$x_2''(t) = x_1'(t) + 2\, x_2(t)\, x_2'(t) + \cos t =$$
$$= x_2(t) + 2\, x_2(t)\, [x_1(t) + (x_2(t))^2 + \sin t] + \cos t$$

hence:

$$G_2(t,x) = \begin{bmatrix} x_1 + x_2^2 + \sin t \\ x_2 + 2\, x_1 x_2 + 2\, x_2^3 + 2\, x_2 \sin t + \cos t \end{bmatrix}$$

(3) The approximation x(k+1) using TS(1) is:

$$x(k+1) = x(k) + F(t(k),x(k))\,h(k) = \begin{bmatrix} x_1(k) + x_2(k)\,h(k) \\ x_2(k) + [x_1(k) + (x_2(k))^2 + \sin t(k)]\,h(k) \end{bmatrix}$$

<u>Problem 4</u>

To approximate the graph of the function:

$$f(x) = \sin 3x$$

on [a,b] = [0,5], using *Scilab* we use the following instructions:

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> x = linspace(0,5,n + 1)';
> plot(x,f(x));
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The effect is to draw, on a Cartesian plane, the graph of $\sigma_n(x)$, the continuous and piecewise linear function on the intervals determined by the points x(1),...,x(n + 1) which interpolates the values of f in x(1),...,x(n + 1).

Find n such that:

$$e_n(f) = \max_{x\in[0,5]} |\sigma_n(x) - f(x)| \leqslant 10^{-2}$$

<u>Solution</u>.

The function f has a continuous second derivative: $f''(x) = -9\sin 3x$. For each $x \in [x(k),x(k+1)]$ we then have (using Theorem (3.11) of Lecture 25):

$$|\sigma_n(x) - f(x)| \leqslant \frac{M_2}{2}|x - x(k)||x - x(k+1)| \quad \text{where} \quad M_2 = \max_{x\in[0,5]}|f''(x)| = 9$$

hence:

$$\max_{x\in[x(k),x(k+1)]}|\sigma_n(x) - f(x)| \leqslant \frac{M_2}{2}\max_{x\in[x(k),x(k+1)]}|x - x(k)||x - x(k+1)|$$

Moreover:

$$\max_{x\in[x(k),x(k+1)]}|x - x(k)||x - x(k+1)| = \left(\frac{x(k+1) - x(k)}{2}\right)^2$$

hence:

$$\max_{x\in[x(k),x(k+1)]}|\sigma_n(x) - f(x)| \leqslant \frac{M_2}{8}[\,x(k+1) - x(k)\,]^2 = \frac{M_2}{8}\left(\frac{b - a}{n}\right)^2$$

Finally we obtain:

$$e_n(f) = \max_{x\in[0,5]}|\sigma_n(x) - f(x)| \leqslant \frac{M_2}{8}\left(\frac{b - a}{n}\right)^2$$

To get $e_n(f) \leqslant 10^{-2}$ it is sufficient to find n such that:

$$\frac{M_2}{8}\left(\frac{b - a}{n}\right)^2 \leqslant 10^{-2} \qquad \text{i.e.} \qquad n \geqslant 10\sqrt{\frac{M_2}{8}}\,(b - a) = 53.03\cdots$$

Hence: $n \geqslant 54$.