Lecture 25 (hrs. 50,51) - November 20, 2025, 8:30 - 10:30 F3

(3.08) Definition (continuous and piecewise linear functions).

Given *ordered* real numbers $x_0 < x_1 < \cdots < x_k$, for $j = 1,...,k$ let $I_j = [x_{j-1},x_j]$. A function $f: [x_0,x_k] \to R$ is said to be *continuous and piecewise linear on* $x_0,...,x_k$ if:

- f is a continuous function;
- f is piecewise linear on $x_0,...,x_k$ i.e.: defined $p_j(x)$ the restriction of f to I_j, it is: $p_j \in P_1(R)$.

The set of continuous and piecewise linear functions on $x_0,...,x_k$ is denoted by C-LAT($x_0,...,x_k$).

(3.09) Example.

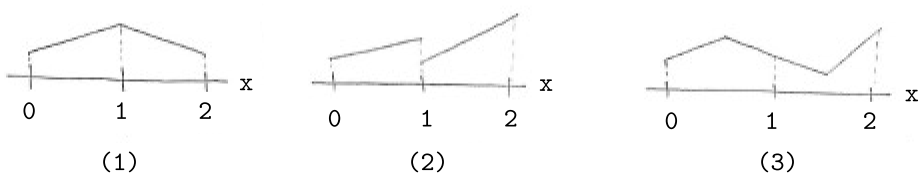


Figure (1) represents the graph of a continuous and piecewise linear function on 0,1,2.
Figure (2) represents the graph of a piecewise linear function on 0,1,2 but *not* continuous.
Figure (3) represents the graph of a continuous and piecewise linear function but *not* on 0,1,2.La figura (1) rappresenta il grafico di una funzione continua e lineare a tratti su 0,1,2.

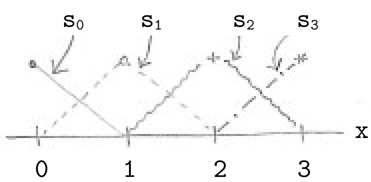
(3.09) Remark.[1]

(1) The set C-LAT($x_0,...,x_k$) is a *vector subspace* of the space of continuous functions on $[x_0,x_k]$, of *dimension* k+1. The k+1 elements $s_0(x),...,s_k(x)$ of C-LAT(x0,...,xk) defined by:

$$s_i(x_j) = 1 \text{ if } i = j \quad , \quad s_i(x_j) = 0 \text{ if } i \neq j$$

are the '*canonical basis*' of C-LAT($x_0,...,x_k$).

For example, the graphs of the elements of the canonical basis of C-LAT(0,1,2,3) are:



(2) Given real numbers $y_0,...,y_k$, the linear combination $y_0\, s_0(x) + \cdots + y_k\, s_k(x)$ is the *unique* element of C-LAT($x_0,...,x_k$) that interpolates the data $(x_0,y_0),...,(x_k,y_k)$.

1 The proof of the statements of this Remark is omitted.

(3.10) Applications.

(1) How we plot graphs in *Scilab*.

Let f: [a,b] → R be a continuous function. The sequence of instructions:

```
> x = linspace(a,b,n)';
> plot(x,f(x));
```

generates, in a graphics window, the graph of the broken line whose vertices are the points with coordinates (x(1),f(x(1)),...,(x(n),f(x(n)). This broken line is the graph of the unique element $\sigma_n(x) \in$ C-LAT(x(1),...,x(n)) that interpolates the data (x(1),f(x(1)),...,(x(n),f(x(n)). The graph of $\sigma_n(x)$ is used as an approximation to the graph of the function f(x). We will see shortly how accurate this approximation is.

(2) Trapezoid formula.

Let f: [a,b] → R be a continuous function. We want to know the value (which certainly exists due to the continuity of f):

$$I = \int_a^b f(x)dx$$

A procedure that provides an approximation of I is the following:

- given an integer number k, the interval [a,b] is divided into k subintervals of equal size:

$$\frac{b-a}{k}$$

  identified by the k+1 points $x_0 = a, x_1, ..., x_{k-1}, x_k = b$ (called *nodes*);
- we consider the only element $\sigma_k(x) \in$ C-LAT$(x_0,...,x_k)$ that interpolates the data $(x_0,f(x_0)),...,(x_k,f(x_k))$;
- we approximate I with:

$$J_k = \int_a^b \sigma_k(x)dx$$

The value J_k is easily computed. By introducing the canonical basis $s_0(x),...,s_k(x)$ of C-LAT$(x_0,...,x_k)$ we have:

$$J_k = \int_a^b \sigma_k(x)dx = \int_a^b (f(x_0)s_0(x) + \cdots + f(x_k)s_k(x))dx = f(x_0)\int_a^b s_0(x)dx + \cdots + f(x_k)\int_a^b s_k(x)dx$$

hence:

$$J_k = h\left[\frac{f(x_0)}{2} + f(x_1) + \cdots + f(x_{k-1}) + \frac{f(x_k)}{2}\right]$$

This last expression is called the *trapezoid formula*. We'll see shortly how accurate this approximation is.

(3.11) Theorem (error in polynomial interpolation).

Given real numbers $x_0 < x_1 < \cdots < x_k$, and set $I = [x_0,x_k]$, let $f: I \to R$ be function with a continuous derivative of order k+1, and let $p_k \in P_k(R)$ be the polynomial that interpolates the data $(x_0,f(x_0)),\dots,(x_k,f(x_k))$. Then, for each $x \in I$, there exists $\theta \in I$ such that:

$$f(x) - p_k(x) = \frac{f^{(k+1)}(\theta)}{(k+1)!}(x-x_0)\cdots(x-x_k)$$

Moreover, defined:

$$M_j = \max_{x \in I} |f^{(j)}(x)|$$

we easily get the limitation:

$$\max_{x \in I} |f(x) - p_k(x)| \leqslant \frac{M_{k+1}}{(k+1)!}(\text{meas } I)^{k+1}$$

Proof: omitted.

(3.12) Remark (approximation with C-LAT elements).

Let $f: [a,b] \to R$ with continuous second derivative and, for $j = 0,\dots,k$, let:

$$x_j = a + \frac{b-a}{k} j$$

The points $x_0,\dots,x_k$ divide the interval [a,b] into k intervals of equal width.

Consider the interval $[x_0,x_1]$. Let $p_1 \in P1(R)$ be the polynomial that interpolates the data $(x_0,f(x_0)),(x_1,f(x_1))$. Using the limitation shown in the previous theorem, we have:

$$\max_{x \in [x_0,x_1]} |f(x) - p(x)| \leqslant \frac{M_2}{2}\left(\frac{b-a}{k}\right)^2$$

In the same way, we see that the same limitation holds for each of the k subintervals of [a,b]. Therefore, denoted by $\sigma_k(x)$ the element of C-LAT$(x_0,\dots,x_k)$ that interpolates the data $(x_0,f(x_0)),\dots,(x_k,f(x_k))$, we have:

$$\max_{x \in [a,b]} |f(x) - \sigma_k(x)| \leqslant \frac{M_2}{2}\left(\frac{b-a}{k}\right)^2$$

(3.13) Remark (accuracy of approximations in Applications (3.10)).

(1) We chose the following quantity as an (absolute) measure of the error made by approximating the graph of f(x) with that of $\sigma_n(x)$:

$$e_n(f) = \max_{x \in [a,b]} |f(x) - \sigma_n(x)|$$

The result of the previous remark shows that: if f has a continuous second derivative, then:

$$\lim_{n\to\infty} e_n(f) = 0$$

and the approximation can be made *as accurate as desired* by choosing an appropriately large number n.

(2) We choose the quantity $|J_k - I|$ as the (absolute) measure of the error made by approximating I with J_k. The result of the previous remark shows that: if f has a second derivative, then:

$$|J_k - I| = \left|\int_a^b (\sigma_k(x)-f(x))\,dx\right| \leqslant \int_a^b |\sigma_k(x)-f(x)|\,dx \leqslant \int_a^b \frac{M_2}{2}\left(\frac{b-a}{k}\right)^2 dx = \frac{M_2}{2}\frac{(b-a)^3}{k^2}$$

In this case too we have:

$$\lim_{k\to\infty} |J_k - I| = 0$$

and the approximation can be made *as accurate as desired* by choosing an appropriately large k.

(3.14) <u>Remark</u> (reformulation of the least squares problem).

Consider the data (k = 2): $(x_0,y_0),(x_1,y_1),(x_2,y_2)$ and the vector space $F = \mathrm{span}\{ f_1(x),f_2(x) \} = \{ a_1 f_1(x) + a_2 f_2(x)$ with $a_1, a_2 \in R \}$. The square deviation SQ(f) is rewritten, using the expression for f(x) in terms of the generators $f_1(x),f_2(x)$, as:

$$SQ(f) = (a_1 f_1(x_0) + a_2 f_2(x_0) - y_0)^2 + (a_1 f_1(x_1) + a_2 f_2(x_1) - y_1)^2 + (a_1 f_1(x_2) + a_2 f_2(x_2) - y_2)^2$$

Observing that if $v = (v_1,\dots,v_n)^t \in R^n$ then we have $(N_2(v))^2 = v_1^2 + \cdots + v_n^2$, the last sum can be rewritten as:[2]

$$\left\|\begin{bmatrix} a_1 f_1(x_0) + a_2 f_2(x_0) - y_0 \\ a_1 f_1(x_1) + a_2 f_2(x_1) - y_1 \\ a_1 f_1(x_2) + a_2 f_2(x_2) - y_2 \end{bmatrix}\right\|^2 = \left\|\begin{bmatrix} f_1(x_0) & f_2(x_0) \\ f_1(x_1) & f_2(x_1) \\ f_1(x_2) & f_2(x_2) \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}\right\|^2$$

hence, defined:

$$A = \begin{bmatrix} f_1(x_0) & f_2(x_0) \\ f_1(x_1) & f_2(x_1) \\ f_1(x_2) & f_2(x_2) \end{bmatrix} \quad , \quad x = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad e \quad b = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}$$

we finally get:

$$SQ(f) = \| A x - b \|^2$$

(3.15) <u>Remark</u>.

The system $A x = b$ obtained in the previous example is *the system that translates the interpolation conditions* $f(x_0) = y_0$, $f(x_1) = y_1$, $f(x_2) = y_2$.

---

2 To simplify the notation, for each $v \in R^n$, in this part we will indicate by $\| v \|$ the two norm of v.