Lecture 24 (hrs. 48,49) - November 19, 2025, 11:30 - 13:30 A13

(2.83) Remark (halt condition).

Let $A \in R^{n \times n}$ be an invertible matrix and $b \in R^n$ be a nonzero column. We use the convergent iterative method defined by $H \in R^{n \times n}$ and $c \in R^n$ to approximate the solution x^* of the system $A\,x = b$. For every $g \in R^n$, the iterative method generates the sequence $x(k)$, convergent to x^*. We describe two possible halt conditions.

(a) Given $E > 0$, let $r(k) = b - A\,x(k)$ (residual vector associated with $x(k)$) and set:

$$\underline{\text{if}}\ \|r(k)\| / \|b\| < E\ \underline{\text{then}}\ \text{STOP}$$

- The condition *is computable*;
- The condition *is effective* (indeed: if $x(k) \to x^*$ then $x(k) - x^* \to 0$ and therefore $r(k) = A\,(x^* - x(k)) \to 0$);
- When the condition is verified, we have, interpreting $x(k)$ as a solution of the perturbed system $A\,x = b - r(k)$ and using the results of the conditioning theory:

$$\|x(k) - x^*\| / \|x^*\| \leqslant c(A)\,\|r(k)\| / \|b\| < c(A)\,E$$

  We get a limitation on the *relative error*. Note that if the conditioning number of A is very large, the returned approximation may not be as accurate as required by the user.

(b) Given $E > 0$ set:

$$\underline{\text{if}}\ \|x(k) - x(k-1)\| < E\ \underline{\text{then}}\ \text{STOP}$$

- The condition *is computable*;
- The condition *is effective* (indeed: if $x(k) \to x^*$ then $x(k-1) - x^* \to 0$ hence $x(k) - x(k-1) \to 0$);
- When the condition is verified, we have: if $\|H\| < 1$ then, defined $F(H) = \|H\| / (1 - \|H\|)$ it is:[1]

$$\|x(k) - x^*\| \leqslant F(H)\,\|x(k) - x(k-1)\| < F(H)\,E$$

  We get a limitation on the *absolute error*. Note that if $\|H\|$ is slightly less than one then $F(H)$ is very large and the returned approximation may not be as accurate as required by the user.

(2.84) Homework.

Write a *Scilab* function, whose header should be

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x = GaussSeidel(A,b,E)
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such that: it verifies that the elements on the diagonal of A are all different from zero and applies the Gauss-Seidel method to the system $A\,x = b$ using as a stopping criterion the one given in (b) of Remark (2.83).

1 Proof omitted.

(3) INTERPOLATION AND LEAST SQUARES

(3.01) Problem.

Let:

$$(x_0,y_0) \ , \ \dots \ , \ (x_k,y_k)$$

be k+1 pairs of real numbers (called *data*) where $x_0,\dots,x_k$ are *pairwise distinct*, and let F be a vector subspace of the vector space on R of the continuous functions from I ⊂ R into R such that:

$$\dim F = m$$

- The *interpolation problem* consists in determining the elements $g \in F$ such that:

  $$g(x_0) = y_0 \ , \ \dots \ , \ g(x_k) = y_k$$

  Each of the elements g that satisfies the conditions is called an element of F *that interpolates the data*.

  The condition that $x_0,\dots,x_k$ are pairwise distinct is necessary for the interpolation problem to have *at least one* solution.

- Let $m < k+1$. The *least squares problem* consists in determining the *absolute minimum points* of the function SQ: F → R defined by:

  $$SQ(f) = (f(x_0) - y_0)^2 + \dots + (f(x_k) - y_k)^2$$

  Each of the elements g that satisfies the condition is called *an element of* F *that best fits the data in the least squares sense.*

  Note that in this problem the condition that $x_0,\dots,x_k$ be pairwise distinct is *not required.*

(3.02) Remark (geometric interpretation of the two problems).

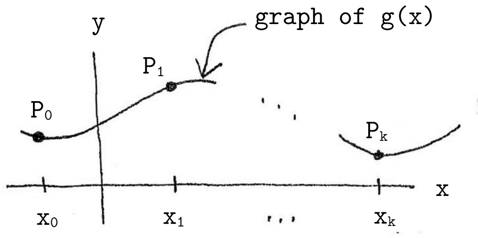
Represent the k+1 points:[2]

$$P_0 \equiv (x_0,y_0), \ \dots \ , P_k \equiv (x_k,y_k)$$

on a Cartesian plane.

The interpolation problem is to *determine the elements* $g \in F$ *whose graph contains all* k+1 *points* (see the following figure).

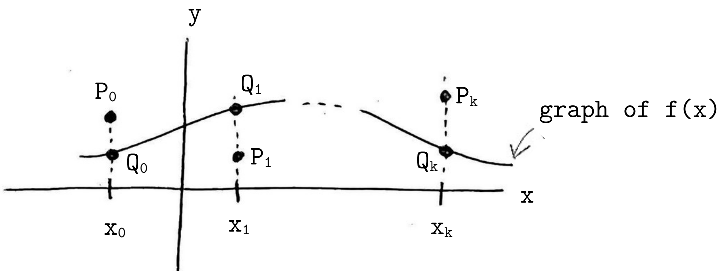
2 Given a Cartesian plane π, by $P \equiv (x,y)$ we mean that the pair of real numbers (x,y) constitutes the *coordinates* of the point $P \in \pi$.



For each $f \in F$ let:

$$Q_0 \equiv (x_0, f(x_0)), \; \ldots \; , Q_k \equiv (x_k, f(x_k))$$

The value SQ(f) is *the sum of the squares of the lengths of the segments* P_0Q_0, ... ,P_kQ_k. This value can be thought of as the 'distance' of the graph of f from the data (see the next figure).



(3.03) Example (reformulation of the interpolation problem).

Consider the data (k = 2): $(x_0,y_0),(x_1,y_1),(x_2,y_2)$, where x_0,x_1,x_2 are pairwise distinct, and the vector space F = span{ $f_1(x), f_2(x)$ } = { $a_1 f_1(x) + a_2 f_2(x)$ with $a_1, a_2 \in R$ }. Using the expression of g(x) in terms of the generators $f_1(x), f_2(x)$, we may rewrite the interpolation conditions:

$$g(x_0) = y_0 \; , \; g(x_1) = y_1 \; , \; g(x_2) = y_2$$

as:

$$a_1 f_1(x_0) + a_2 f_2(x_0) = y_0 \; , \; a_1 f_1(x_1) + a_2 f_2(x_1) = y_1 \; , \; a_1 f_1(x_2) + a_2 f_2(x_2) = y_2$$

Hence:

$$g(x) = a_1 f_1(x) + a_2 f_2(x) \text{ interpolates the data}$$

$$\Updownarrow$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \text{ is a solution of the system } \begin{bmatrix} f_1(x_0) & f_2(x_0) \\ f_1(x_1) & f_2(x_1) \\ f_1(x_2) & f_2(x_2) \end{bmatrix} z = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}$$

The system has *as many equations as there are data to be interpolated*, and *as many unknowns as there are assigned generators of* F.

(3.04) Remark (polynomial interpolation).

Let: $(x_0,y_0),\ldots,(x_k,y_k)$, where $x_0,\ldots,x_k$ are pairwise distinct, and let F = $P_k(R)$.[3] Note that

3 $P_k(R)$ is the vector space on R of polynomials with real coefficients of degree *at most* k.

in this case the size of F is *equal* to the number of data points. The problem of determining the elements of $P_k(R)$ that interpolate the data is called *the polynomial interpolation problem*. To study the problem, a basis is introduced in $P_k(R)$. The choice of basis affects the form of the system to be solved and the expression of any determined elements of F. Let's look at three possible choices.

(1) Consider the *Vandermonde basis* of $P_k(R)$:

$$1, x, x^2, \dots, x^k$$

The system that translates the interpolation conditions is:

$$\begin{bmatrix} 1 & x_0 & \dots & x_0^k \\ \vdots & \vdots & & \vdots \\ 1 & x_k & \dots & x_k^k \end{bmatrix} c = \begin{bmatrix} y_0 \\ \vdots \\ y_k \end{bmatrix}$$

whose matrix is called the *Vandermonde matrix*. If $c^* = (c_0, \dots, c_k)^t$ is a solution of the system, the polynomial:

$$p_k(x) = c_0 + c_1 x + \cdots + c_k x^k$$

interpolates the data and the resulting expression is called the *Vandermonde form* of the polynomial.

(2) Consider the *Newton basis* of $P_k(R)$:

$$1, (x - x_0), (x - x_0)(x - x_1), \dots, (x - x_0)\cdots(x - x_{k-1})$$

It is easily verified that the system that translates the interpolation conditions, for example in the case k = 3, is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & x_1 - x_0 & 0 & 0 \\ 1 & x_2 - x_0 & (x_2 - x_0)(x_2 - x_1) & 0 \\ 1 & x_3 - x_0 & (x_3 - x_0)(x_3 - x_1) & (x_3 - x_0)(x_3 - x_1)(x_3 - x_2) \end{bmatrix} c = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

The matrix of the system is *lower triangular* and invertible (remember that the numbers $x_0, \dots, x_k$ are pairwise distinct). The Newton basis is *specifically constructed* so that the matrix of the system is lower triangular. If $c^* = (c_0, \dots, c_k)^t$ is a solution to the system, the polynomial:

$$p_k(x) = c_0 + c_1 (x - x_0) + \cdots + c_k (x - x_0)\cdots(x - x_{k-1})$$

interpolates the data and the resulting expression is called the *Newton form* of the polynomial.

(3) Consider the k+1 elements of $P_k(R)$ defined by:

$$l_0(x) = \frac{(x - x_1)\cdots(x - x_k)}{(x_0 - x_1)\cdots(x_0 - x_k)}, \; l_1(x) = \frac{(x - x_0)(x - x_2)\cdots(x - x_k)}{(x_1 - x_0)(x_1 - x_2)\cdots(x_1 - x_k)}, \dots, \; l_k(x) = \frac{(x - x_0)\cdots(x - x_{k-1})}{(x_k - x_0)\cdots(x_k - x_{k-1})}$$

These elements are constructed in such a way that for i = 0,...,k we have: $l_i(x_i) = 1$ and $l_i(x_j) = 0$ for $j \neq i$. Furthermore, they are *linearly independent* elements of $P_k(R)$ (indeed: *if* $A(x) = a_0\, l_0(x) + \cdots + a_k\, l_k(x) = 0$ for every $x \in R$ then for i = 0,...,k we have: $A(x_i) = a_i = 0$) and therefore, since dim $P_k(R) = k+1$, they are a *basis* of Pk(R), called *the Lagrange basis* of $P_k(R)$.

It is easily verified that the system that translates the interpolation conditions is:

$$c = \begin{bmatrix} y_0 \\ \vdots \\ y_k \end{bmatrix}$$

Indeed, the system matrix is the *identity matrix*. The Lagrange basis is *specifically constructed* to make this happen. Finally, the polynomial:

$$p_k(x) = y_0\, l_0(x) + \cdots + y_k\, l_k(x)$$

interpolates the data and the resulting expression is called the *Lagrange form* of the polynomial.

(3.05) Theorem (existence and uniqueness of the interpolating polynomial)

Let: $(x_0,y_0),\ldots,(x_k,y_k)$, where $x_0,\ldots,x_k$ are pairwise distinct. There is *only one* element $p(x) \in P_k(R)$ that interpolates them. The polynomial p(x) is called *the interpolating polynomial*.

(Proof. As shown in point (3) of the previous Remark, there is only one linear combination of the elements of the Lagrange basis that interpolates the data. Therefore there is only one element of $P_k(R)$ that interpolates the data.)

(3.06) Remark.

To solve a polynomial interpolation problem, a basis of $P_k(R)$ is *chosen*, *the* solution to the system that translates the interpolation conditions is determined, and *the* linear combination of the elements of the chosen basis that interpolates the data is identified. Depending on the chosen basis, *a different form of the unique interpolating polynomial* is obtained.

(3.07) Homework.

Solve the following problems, *none of which are polynomial interpolation problems* (why?).

(1) Let (-1,1), (0,0), (1,0) be the data. Find the elements $g \in P_1(R)$ that interpolate the data.
(2) Let (-1,0), (0,0), (1,0) be the data. Find the elements $g \in P_1(R)$ that interpolate the data.
(3) Let (0,0) be the data. Find the elements $g \in P_1(R)$ that interpolate the data.