Lecture 23 (hrs. 45,46,47) - November 18, 2025, 15:30 - 18:30 F3

(2.68) Definition (Jacobi method).

Let $A \in R^{n\times n}$ be an invertible matrix with $A(k,k) \neq 0$ for every k. Defined:[1]

$$D = \text{diag}(A) \quad , \quad M = A - D$$

the matrix D is invertible and the system $A\,x = b$ is equivalent to $x = -D^{-1} M\,x + D^{-1} b$.
The *Jacobi method* (applied to the system $A\,x = b$) is the iterative method defined by:
$H_J = -D^{-1} M$ and $c_J = D^{-1} b$.

(2.69) Definition (strictly diagonally dominant matrix).

Let $A \in R^{n\times n}$. The matrix A is *strictly row-diagonally dominant* if

$$\text{for every } k: \quad |A(k,k)| > \sum_{i\neq k} |A(k,i)|$$

(2.70) Theorem (strictly diagonally dominant ⇒ invertible).

Let $A \in R^{n\times n}$. If A is strictly row-diagonally dominant then A is invertible.

(Proof: By contradiction, if A were strongly row-diagonally dominant and non-invertible then there would exist a column $y \neq 0$ such that $A\,y = 0$. Let y_j be the component of y of maximum modulus (certainly different from zero). Then it would be:

$$A(j,1)\,y_1 + \ldots + A(j,j)\,y_j + \ldots + A(j,n)\,y_n = 0 \quad \text{i.e.} \quad A(j,j)\,y_j = -\sum_{i\neq j} A(j,i)y_i$$

hence:

$$|A(j,j)\,y_j| = \left|\sum_{i\neq j} A(j,i)y_i\right| \;\Rightarrow\; |A(j,j)|\,|y_j| \leqslant \sum_{i\neq j} |A(j,i)||y_i|$$

Since, by definition, $y_j \neq 0$ and for every $i \neq j$ it is $|y_i| / |y_j| \leqslant 1$ finally it would be:

$$|A(j,j)| \leqslant \sum_{i\neq j} |A(j,i)| \left|\frac{y_i}{y_j}\right| \leqslant \sum_{i\neq j} |A(j,i)|$$

i.e. a contradiction.)

(2.71) Example.

Let:[2]

$$A = \begin{bmatrix} 3 & & & 1 \\ 1 & 3 & & 1 \\ 1 & & 3 & 1 \\ 1 & & & 3 \end{bmatrix} \quad , \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

1 If $A \in R^{n\times n}$, diag(A) denotes the matrix [A(1,1);...;A(n,n)]. The notation is borrowed from *Scilab*.

2 If an element is *not specified* when writing an array, its value is *zero*.

- The matrix A is strictly row-diagonally dominant and therefore invertible, and has all diagonal entries nonzero. The Jacobi method is defined and we have:

$$H_J = -\frac{1}{3}\begin{bmatrix} 0 & & & 1 \\ 1 & 0 & & 1 \\ 1 & & 0 & 1 \\ 1 & & & 0 \end{bmatrix} \quad , \quad c_J = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- The eigenvalues of H_J ($\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 1/3$, $\lambda_4 = -1/3$) all have modulus less than one. By the Characterization Theorem (2.66) of Lecture 22, the method is convergent. For every $g \in R^4$, the sequence generated by the method starting from g is convergent to the solution x^* of the system A x = b.

(2.72) Theorem (sufficient convergence condition for the Jacobi method).

Let $A \in R^{n \times n}$ be a strictly row-diagonally dominant matrix and let $b \in R^n$. Then the Jacobi method applied to the system A x = b is convergent.

The result is a simple consequence of the following theorem and observation.

(2.73) Theorem (norm and spectral radius).

Let $A \in R^{n \times n}$ and let N be a norm in R^n. Then: $\rho(A) \leqslant \| A \|_N$.

Proof. By definition: $\| A \|_N = \max\{ N(A\,v),\ N(v) = 1 \}$. Then, let $\lambda \in C$ be an eigenvalue of A and $w \in R^n$ be an associated eigenvector. Then, setting $w' = w / N(w)$ we have:

$$N(w') = 1 \quad \text{and} \quad N(A\,w') = N(A\frac{w}{N(w)}) = \frac{N(A\,w)}{N(w)} = \frac{N(\lambda\,w)}{N(w)} = |\lambda|\frac{N(w)}{N(w)} = |\lambda|$$

hence $|\lambda| \in \{ N(A\,v),\ N(v) = 1 \}$. Finally:

$$\rho(A) = \max\{ |\lambda| \text{ s.t. } \lambda \in \sigma(A) \} \leqslant \max\{ N(A\,v),\ N(v) = 1 \} = \| A \|_N$$

(2.74) Remark.

Let $A \in R^{n \times n}$ and- let $b \in R^n$. If A is strictly row-diagonally dominant then for the H_J matrix of the Jacobi method applied to the system A x = b we have $\| H_J \|_\infty < 1$.

(Homework: Prove that the statement is an immediate consequence of the definition of strictly row-diagonally dominant matrix.)

(2.75) Example (*Scilab*).

Consider Example (2.71). To construct the matrix A in *Scilab*, the following assignments are used:[3]

3 In *Scilab*: for any integer n, eye(n,n) is the identity matrix of order n; if A is a matrix and m,k,l are integers then: A(m:k,l) = [A(m,l);...;A(k,l)].

```
--> A = 3 * eye(4,4)

 A = [4x4 double]

   3.   0.   0.   0.
   0.   3.   0.   0.
   0.   0.   3.   0.
   0.   0.   0.   3.

--> A(2:4,1) = 1

 A = [4x4 double]

   3.   0.   0.   0.
   1.   3.   0.   0.
   1.   0.   3.   0.
   1.   0.   0.   3.

--> A(1:3,4) = 1

 A = [4x4 double]

   3.   0.   0.   1.
   1.   3.   0.   1.
   1.   0.   3.   1.
   1.   0.   0.   3.

--> b = [1;1;1;1]

 b = [4x1 double]

   1.
   1.
   1.
   1.
```

To construct the H_J matrix and the c_J column:[4]

```
--> D = diag(diag(A))

 D = [4x4 double]

   3.   0.   0.   0.
   0.   3.   0.   0.
   0.   0.   3.   0.
   0.   0.   0.   3.
```

4 In *Scilab*, if A is an n $\times$ n matrix then diag(A) = [A(1,1);...;A(n,n)]; if v = $[v_1;...;v_n] \in R^n$ then diag(v) is the diagonal matrix $M \in R^{n\times n}$ such that M(1,1) = v_1,..., M(n,n) = v_n.

```
--> M = A - D

 M = [4x4 double]

   0.   0.   0.   1.
   1.   0.   0.   1.
   1.   0.   0.   1.
   1.   0.   0.   0.

--> HJ = - diag(1./diag(A)) * M

 HJ = [4x4 double]

   0.          0.   0.  -0.3333333
  -0.3333333   0.   0.  -0.3333333
  -0.3333333   0.   0.  -0.3333333
  -0.3333333   0.   0.   0.

--> cJ = diag(1./diag(A)) * b

 cJ = [4x1 double]

   0.3333333
   0.3333333
   0.3333333
   0.3333333
```

An approximation to the solution of the system A x = b, computed using the built-in *backslash* (\) function[5] is:

```
--> y = A\b

 y = [4x1 double]

   0.25
   0.1666667
   0.1666667
   0.2500000
```

To obtain an approximation of the solution with the Jacobi method, ten elements of the sequence generated by the method are calculated starting from the vector 0.[6] At each iteration the instruction disp(norm(x - y,%inf)) shows the value $\| x - y \|_\infty$ or the distance between the last calculated element, x, of the sequence and y.

```
--> x = zeros(4,1); for k = 1:10, x = HJ * x + cJ; disp(norm(x - y,%inf)); end
```

5 The assignment y = A\b is equivalent to the sequence: (S,D,P) = $EGPP_M$(A); w = SA_M(S,P b); y = SI_M(D,w).

6 If m,n are integers, zeros(m,n) is the matrix of order m × n whose elements are all equal to zero.

0.1666667

0.0555556

0.0185185

0.0061728

0.0020576

0.0006859

0.0002286

0.0000762

0.0000254

0.0000085

Note that, as expected by the convergence of the sequence, the distance $\| x - y \|_\infty$ is decreasing.

(2.76) <u>Definition</u> (Gauss-Seidel method).

Let $A \in R^{n \times n}$ be an invertible matrix with $A(k,k) \neq 0$ for every k. Defined:[7]

$$T = \text{tril}(A) \ , \quad M = A - T$$

the matrix T is invertible and the system $A x = b$ is equivalent to $x = - T^{-1} M x + T^{-1} b$. The *Gauss-Seidel method* (applied to the system $A x = b$) is the iterative method defined by: $H_{GS} = - T^{-1} M$ and $c_{GS} = T^{-1} b$.

(2.77) <u>Example</u>.

Let A and b be as in Example (2.71).

- The matrix A is strictly row-diagonally dominant and therefore invertible, and has all diagonal entries nonzero. The Gauss-Seidel method is defined and we have:

$$H_{GS} = \begin{bmatrix} & & & -1/3 \\ & & & -2/9 \\ & & & -2/9 \\ & & & 1/9 \end{bmatrix} \in R^{4 \times 4} \quad , \quad c_{GS} = \begin{bmatrix} 1/3 \\ 2/9 \\ 2/9 \\ 2/9 \end{bmatrix}$$

- The eigenvalues of H_{GS} ($\lambda_1 = \lambda_2 = \lambda_3 = 0$, $\lambda_4 = 1/9$) all have modulus less than one. By the Characterization Theorem (2.66) of Lecture 22 the method is convergent. For every $g \in R4$ the sequence generated by the method starting from g is convergent to

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7 If $A \in R^{n \times n}$, we denote by tril(A) the strictly lower triangular part of A, that is, the (lower triangular) matrix B such that: $i \leqslant j \Rightarrow B(i,j) = A(i,j)$ and $i > j \Rightarrow B(i,j) = 0$. The notation is borrowed from *Scilab*.

the solution x^* of the system $A x = b$.

(2.78) Theorem (sufficient convergence condition for the Gauss-Seidel method).

Let $A \in R^{n \times n}$ and $b \in R^n$. If:

(1) A is a strictly row-diagonally dominant matrix

or:

(2) A a symmetric positive-definite matrix

then the Gauss-Seidel method applied to the system $A x = b$ is convergent.

(2.4) COST OF SOLVING A SYSTEM OF LINEAR EQUATIONS WITH AN ITERATIVE METHOD

(2.79) Remark.

Let $A \in R^{n \times n}$, $b \in R^n$ and let x', x" be the approximations of the solution x^* of the system $A x = b$ obtained, respectively, with a *direct method* and with an *iterative method* (equipped, as we will see, with an appropriate halt condition). We want to compare x' and x" from the point of view of *arithmetic cost*.

Assume x' is calculated by using the sequence EGPP-SA-SI. The asymptotic cost of the computation is then: $(2/3)\, n^3$.

The cost of computing x" is:

(cost of one iteration) * (number of iterations)

We then need to determine the cost of *one* iteration.

Consider, for example, the Gauss-Seidel method. To compute the column x(k+1), there are (at least) two alternatives:

(1) compute $- T^{-1} M\, x(k) + T^{-1} b$;
(2) compute the solution of the system $T x = - M x(k) + b$.

For the cost of the first alternative we have:

(1.a) $2 n^2 - 3 n$ operations to compute $- T^{-1} M\, x(k)$
(1.b) n operations to compute the sum $- T^{-1} M\, x(k) + T^{-1} b$

hence a total of $2 n^2 - 2 n$ operations.

For the cost of the second alternative we have:

(2.a) $n^2 - 2 n + 1$ operations to compute $- M\, x(k)$

(2.b) n operations to compute the sum $- M\,x(k) + b$
(2.c) n^2 operations to compute the solution of the system

hence a total of $2\,n^2 - n + 1$ operations.

In both cases, the *asymptotic cost* is $2\,n^2$. Therefore, if x" was calculated with k iterations of the Gauss-Seidel method, the asymptotic cost of the calculation is $2\,k\,n^2$. The Gauss-Seidel method is cheaper than the direct method if $k < n/3$.

(Homework: check the costs for both alternatives.)

It is necessary to study the *speed of convergence* of an iterative method.

(2.80) Example.

Let $H = \mathrm{diag}(s_1, s_2)$ with $|s_2| < |s_1| < 1$ and let $c = 0$. By the Characterization Theorem of Convergent Methods (see Theorem (2.66) of Lecture 22), the iterative method defined by H and zero is convergent: for every $g \in R^2$ the sequence $x(k)$ converges to zero. How quickly?

Let:

$$g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \neq 0.$$

Then:

$$x(k) = H^k g = \mathrm{diag}(s_1^k, s_2^k)\, g = \begin{bmatrix} s_1^k g_1 \\ s_2^k g_2 \end{bmatrix}$$

and, using the one norm:

$$\|x(k)\|_1 = |s_1^k g_1| + |s_2^k g_2|$$

- If $g_1 \neq 0$:

  $$\|x(k)\|_1 = |s_1|^k\,|g_1|\,(1 + |s_2/s_1|^k\,|g_2/g_1|)$$

  hence:

  $$\frac{\|x(k)\|_1}{|s_1|^k} \to |g_1| \neq 0$$

  and:

  $\|x(k)\|_1$ tends to zero *as rapidly as* $|s_1|^k$

- On the contrary, if $g_1 = 0$:

  $$\frac{\|x(k)\|_1}{|s_2|^k} \to |g_2| \neq 0$$

  and:

  $\|x(k)\|_1$ tends to zero *as rapidly as* $|s_2|^k$

  therefore, since $|s_2| < |s_1|$, *more rapidly* than $|s_1|^k$.

(2.81) Theorem (speed of convergence).

What happens in Example (2.80) holds in general.

Consider the convergent iterative method defined by $H \in R^{n \times n}$ and $c \in R^n$. Let x^* be the solution of the system $(I - H)\,x = c$ and let $x(k)$ be the sequence generated by the method starting from $g \in R^n$. Then, denoted by $\rho(H)$ the spectral radius of H:[8]

$$\| x(k) - x^* \| \text{ tends to zero } \textit{at least as quickly as } \rho(H)^k$$

Furthermore, if the initial vector g is chosen randomly, the probability that the sequence converges to zero *more quickly than* $\rho(H)^k$ is zero.

(2.82) Example.

Based on what was obtained in examples (2.71) and (2.77) where $\rho(H_J) = 1/3$ and $\rho(H_{GS}) = 1/9$: with a random choice of $g \in R^2$, the sequence generated by the Gauss-Seidel method converges to x^* *more quickly* than the one generated by the Jacobi method.

8 See Definition (2.65) of Lecture 22.