Lecture 21 (hrs. 41,42) - November 12, 2025, 11:30 - 13:30 A13

(2.55) Remark.

Let's analyze the cost of solving the system $A\,x = b$, with $A \in R^{n\times n}$ an invertible matrix, using the EGPP procedure. The procedures executed are EGPP, SA, and SI. We have:

$$C(EGPP) = \frac{2}{3}n^3 + \ldots \quad , \quad C(SA) = C(SI) = n^2$$

Note that while in the computation of SA and SI *only* arithmetic operations are performed, in the calculation of EGPP *comparisons* are also performed, but their number is *negligible* compared to that of the arithmetic operations.[1]

(Exercise: Determine the number of comparisons performed by EGPP.)

The overall number of arithmetic operation is:

$$C(EGPP) + C(SA) + C(SI) = \frac{2}{3}n^3 + \ldots$$

When we use the qr procedure to solve the system, the qr, matrix by column product (denoted by: pmc) and SI procedures are performed. We have:

$$C(qr) = \frac{4}{3}n^3 + \ldots \quad , \quad C(pmc) = 2\,n^2 + \ldots \quad , \quad C(SI) = n^2$$

Note that in the qr procedure (as in the GS procedure) the computation of square roots is also performed but their number (n) is *negligible* compared to that of the arithmetic operations.

(Exercise: Determine the number of arithmetic operations performed by pmc.)

The overall number of arithmetic operation is:

$$C(qr) + C(pmc) + C(SI) = \frac{4}{3}n^3 + \ldots$$

The dominant term in the arithmetic cost of the procedure using qr is therefore *two times larger* than that of the procedure using EGPP.

1 It can be reasonably assumed that the time needed to compare two machine numbers is *similar* to the time needed to perform an arithmetic operation on the same numbers.

(2.4) ITERATIVE METHODS FOR THE SOLUTION OF A SYSTEM OF LINEAR EQUATIONS

(2.56) Definition (iterative method for the solution of a system of linear equations).

Let $H \in R^{n \times n}$ and $c \in R^n$. *The iterative method defined by* H *and* c is the application that maps each vector $g \in R^n$ to the sequence of vectors x(k) defined by:

$$x(0) = g \quad , \quad x(k) = H\,x(k-1) + c \quad \text{for } k = 1,2,...$$

(2.57) Remark.

- The iterative method defined by H and c is the iterative method defined by the function $h: R^n \to R^n$ such that:
$$h(x) = H\,x + c$$
The function h is *continuous* therefore (see Remark (1.54) in Lecture 8) if the sequence x(k) generated by the method is convergent, then its limit $v \in R^n$ is a *fixed point* of h, that is, it satisfies the relation:
$$v = H\,v + c \quad \text{equivalent to} \quad (I - H)\,v = c$$
and this last relationship means that:
v is the solution of the system of linear equations (I - H) x = c
- Let $A \in R^{n \times n}$ be an invertible matrix. The iterative method defined by H and c *can be used* to approximate the solution of the system A x = b if:
  (1) the systems A x = b and (I - H) x = c are *equivalent* (in particular: I - H is invertible) and
  (2) it is (practically) possible to find $g \in R^n$ from which the sequence generated by the method is convergent.

(2.58) Example.

(1) Let:

$$A = \begin{bmatrix} 1/2 & 0 \\ 0 & -1 \end{bmatrix} \quad , \quad b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Set: $H = I - A = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}$ and c = b; the systems A x = b e (I - H) x = c are equivalent;
- Let $g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$. The sequence generated by the method defined by H and c is then:
$$x(0) = g \quad , \quad x(1) = H\,x(0) + c = H\,g \quad , \quad x(2) = H\,x(1) + c = H^2 g \quad , \quad ...$$
hence:

$$x(k) = H^k g = \begin{bmatrix} (1/2)^k & 0 \\ 0 & 2^k \end{bmatrix} g = \begin{bmatrix} (1/2)^k g_1 \\ 2^k g_2 \end{bmatrix}$$

The sequence is convergent *if and only if* $g_2 = 0$. In this case we have:

$$\lim_{k\to\infty} x(k) = 0$$

and the sequence converges to the unique solution of the system $A x = b$.

(2) Let:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad , \quad b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Set: $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and rewrite $A = 2 I + J$. Then:

  $$A x = b \quad \text{is equivalent to} \quad x = -(1/2) J x + (1/2) b$$

  i.e., defined $H = -(1/2) J$ and $c = (1/2) b$:

  $$A x = b \quad \text{is equivalent to} \quad (I - H) x = c$$

- The eigenvalues of the matrix H are: $\lambda_1 = -1/2$ and $\lambda_2 = 1/2$, hence H is a diagonalizable matrix.[2] It is:

  $$H = S \begin{bmatrix} -1/2 & 0 \\ 0 & 1/2 \end{bmatrix} S^{-1} \quad \text{where} \quad S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

- Set $g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$. The sequence generated by the method defined by H and c is:

  $$x(k) = H^k g = S \begin{bmatrix} (-1/2)^k & 0 \\ 0 & (1/2)^k \end{bmatrix} S^{-1} g$$

  i.e., defined $y = S^{-1} g$:

  $$x(k) = \begin{bmatrix} (-1/2)^k y_1 \\ (1/2)^k y_2 \end{bmatrix}$$

  In this case we have:

  $$\text{for every } g \in R^2 : \lim_{k\to\infty} x(k) = 0$$

  that is: for every $g \in R^2$ the sequence converges to the unique solution of the system $A x = b$.

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2 Recall that (1) a matrix $M \in R^{n \times n}$ is diagonalizable if there exist a diagonal matrix Λ and an invertible matrix S such that: $M S = S \Lambda$, or $M = S \Lambda S^{-1}$; the elements $\lambda_1, \ldots, \lambda_n$ on the diagonal of Λ are the eigenvalues of M, the k-th column of S is an eigenvector associated with the eigenvalue λ_k; (2) if a matrix M has distinct eigenvalues then M is diagonalizable.

(3) Let $A = -I$ and $b = 0$.

- Set $H = I - A$ and $c = b$; the systems $A\,x = b$ e $(I - H)\,x = c$ are equivalent.
- The sequence generated by the iterative method defined by H and c starting from $g \in R^n$ is:

$$x(k) = H^k g = 2^k g$$

The sequence is convergent *if and only if* $g = 0$ and, in this case, it converges to the unique solution of the system $A\,x = b$.