Lecture 19 (hrs. 36,37) - November 6, 2025, 8:30 - 10:30 F3

(2.36) Exercise.

Let V = R^2 with 2-norm and let v $\in R^2$ be such that $\| v \|_2 = 2$.

- Let $x^* = v$. Draw the set of $\hat{x}$ such that $\varepsilon_x \leqslant 1/4$.
- Let $x^* = v/2$. Draw the set of $\hat{x}$ such that $\varepsilon_x \leqslant 1/4$.

(2.37) Exercise.

Let V = R^2 with 2-norm and let v $\in R^2$ be such that $\| v \|_2 = 2$.

- Let $x^* = v$. Draw the set of $\hat{x}$ such that $\| \delta x \|_2 \leqslant 1/2$.
- Let $x^* = v/2$. Draw the set of $\hat{x}$ such that $\| \delta x \|_2 \leqslant 1/2$.

(2.38) Exercise.

In R^2 with 2-norm, let: $x^* = [\ 2;\ 0{,}1]$ and let $\hat{x}$ be such that $\varepsilon_x \leqslant L$. Determine: $\max |\delta x_1 / x^*_1|$ and $\max |\delta x_2 / x^*_2|$.

Solution: $\varepsilon_x \leqslant L \Rightarrow \| \delta x \|_2 \leqslant L \| x^* \|_2$. Hence, for k = 1,2 it is:

$$\max |\delta x_k / x^*_k| = \max |\delta x_k| / |x^*_k| = \max \| \delta x \|_2 / |x^*_k| \leqslant L \| x^* \|_2 / |x^*_k|$$

Then:

$$\max |\delta x_1 / x^*_1| \leqslant L \| x^* \|_2 / |x^*_1| = \text{sqrt}(4 + 0.01) / 2 \approx L$$

and:

$$\max |\delta x_2 / x^*_2| \leqslant L \| x^* \|_2 / |x^*_2| = \text{sqrt}(4 + 0.01) / 0{,}1 \approx 20\ L$$

For the first component, the relative error has a limitation similar to that of the deviation; for the second, however, the limitation is *worse*. This happens because $\| x^* \|_2 / |x^*_1| \approx 1$ but $\| x^* \|_2 / |x^*_2|$ is *much larger* than 1.

(2.39) Remark.

Let $A \in R^{n \times n}$ be an invertible matrix, let $b \in R^n$, let x^* be the solution of the system A x = b and let $\hat{x} \in R^n$. We use $\hat{x}$ to approximate x^*. The question arises as to how accurate the approximation is. Having chosen a norm in R^n, we use the quantity $N(\hat{x} - x^*)/N(x^*)$ to measure the accuracy.

(A) To obtain information on the accuracy, we introduce the *residual vector* of the system A x = b associated with $\hat{x}$ defined by:

$$r = A\,\hat{x} - b$$

and $\hat{x}$ is *interpreted* as the solution of the perturbed system:

$$A\,x = b + r$$

obtained with the perturbations $\delta A = 0$ and $\delta b = r$. With this interpretation of $\hat{x}$ the quantity $N(\hat{x} - x^*)/N(x^*)$ turns out to be the relative measure ε_x of the deviation of the solution due to the perturbation. Applying the Conditioning Theorem (2.36) of Lecture 18 we obtain the limitation:

$$N(\hat{x} - x^*)/N(x^*) = \varepsilon_x \leqslant c_N(A)\, \varepsilon_b \quad \text{con} \quad \varepsilon_b = N(r)/N(b)$$

(B) To obtain information about the accuracy, we look for a matrix $M \in R^{n \times n}$ such that:

$$M\, \hat{x} = -r$$

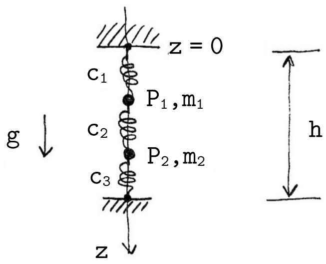
and, given $\delta A = M$, $\hat{x}$ is *interpreted* as the solution of the perturbed system:

$$(A + \delta A)\, x = b$$

With this interpretation of $\hat{x}$ the quantity $N(\hat{x} - x^*)/N(x^*)$ turns out to be the relative measure ε_x of the deviation of the solution due to the perturbation. If $c_N(A)\, \varepsilon_A < 1$, from the Conditioning Theorem (2.36) of Lecture 18 we obtain the limitation:

$$N(\hat{x} - x^*)/N(x^*) = \varepsilon_x \leqslant c_N(A)\, \varepsilon_A \,/\, (1 - c_N(A)\, \varepsilon_A)$$

(2.40) Example.



Consider the system shown in the figure, composed of two heavy points, P_1 of mass m_1, and P_2 of mass m_2, free to slide along a vertical guide and connected by three ideal springs with a rest length of 0, as in the drawing.

Having chosen the descending vertical z-axis, to determine the equilibrium configurations, the static equations are written for each of the points:

$$m_1\, g - c_1\, z_1 + c_2\, (z_2 - z_1) = 0$$
$$m_2\, g - c_2\, (z_2 - z_1) + c_3\, (h - z_2) = 0$$

which, in the form of a system, are rewritten:

$$\begin{bmatrix} c_1 + c_2 , & -c_2 ; \\ -c_2 , & c_2 + c_3 \end{bmatrix} \begin{bmatrix} z_1 ; \\ z_2 \end{bmatrix} = \begin{bmatrix} m_1\, g ; \\ m_2\, g + c_3\, h \end{bmatrix}$$

$$A \qquad\qquad z \quad = \qquad b$$

Using the parameter values:

$$c_1 = c_2 = c_3 = 100 \text{ N/m} \quad , \quad m_1 = m_2 = 1 \text{ kg} \quad , \quad h = 5 \text{ m} \quad , \quad g = 9.81 \text{ m/s}^2$$

the solution z^* of the system is:

$$z_1^* \approx 1.76 \text{ m} \quad , \quad z_2^* \approx 3.43 \text{ m}$$

If we now assume as the value of the gravitational acceleration a value g' such that:[1]

$$|g' - g| = |\,\delta g\,| < 10^{-2}$$

the system $A\,z = b$ is transformed into the perturbed system $A\,z = b + \delta b$ with:

$$\delta b = [\ m_1\,\delta g\ ;\ m_2\,\delta g\ ]$$

Then, choosing the 1-norm in R^2, we have:

$$\varepsilon_b = N_1(\delta b)/N_1(b) < 4 \times 10^{-5} \quad \text{and} \quad c_1(A) = 3$$

According to the Conditioning Theorem, for the deviation of the solution $\hat{z}$ of the perturbed system from the solution z^* we have the following limitation:

$$\varepsilon_x \leqslant c_N(A)\,\varepsilon_b < 1.2 \times 10^{-4}$$

Finally, since:

$$\|\,z^*\,\|_1 \,/\, |z_1^*| \approx 3 \quad \text{and} \quad \|\,z^*\,\|_1 \,/\, |z_2^*| \approx 1.5$$

similar estimates are also obtained for the deviation of the components (see Exercise (2.38)).

1 Remember that the value of the gravitational acceleration is known only approximately.