Lecture 16 (hrs. 29,30) - October 28, 2025, 16:30 - 18:30 F3

(2.06) Remark (simple cases, last part).

(O) *orthogonal* (A is orthogonal if one of the following three *equivalent* conditions subsists:

(1) the columns (or rows) of A are an *orthonormal basis* of R^n with respect to the canonical scalar product;
(2) A is invertible and $A^{-1} = A^t$;
(3) $A^t A = A A^t = I$)

- A is *certainly invertible.*
- The solution x^* of the system $A x = b$ is determined by:

$$x^* = A^t b$$

The number of operations needed to determine the solution is the number of operations needed to perform the product of a matrix by a column:

$$n^2 \textit{ multiplications} + n(n-1) \textit{ sums}$$

(P) *permutation matrix* (A is a permutation matrix if it is obtained from the identity matrix I by *permuting* the column).

The columns of a permutation matrix are therefore those of the identity matrix (except for the order). Therefore, they constitute an orthonormal basis of R^n with respect to the canonical dot product (the canonical basis). It follows that *a permutation matrix is orthogonal.*

- In this case too we have: A is *certainly invertible.*
- The solution x^* of the system $A x = b$ is determined by

$$x^* = A^t b$$

The number of operations needed to determine the solution is, this time, *zero* because A^t, as A, is a permutation matrix and the product $P v$ of a permutation matrix P by a column v produces a column that has *the same components* as v but in a different order.

(2.07) Remark (general case).

When the matrix A of the system does *not* have a structure that allows it to fall into a simple case, the problem is faced in two steps:

Step one:

We *factor* A into a product of *simple factors.*

Example: $A = F_1 F_2 F_3$, with F_1 orthogonal, F_2 upper triangular and F_3 a permutation matrix.

Step two:

Factorization is used to decide whether A is invertible and, if so, to determine the solution x^*.

Example:

$$A = F_1 F_2 F_3 \Rightarrow \det A = \det F_1 \ \det F_2 \ \det F_3$$

hence: A *is invertibile* $\Leftrightarrow$ *each one of the factors is invertible*. Then:

(1) $$A\,x = b \quad \equiv \quad F_1 F_2 F_3\, x = b \quad \equiv \quad F_2 F_3\, x = F_1^{-1}\, b = c_1$$

and we find c_1 as the solution of the simple system $F_1\, x = b$.

(2) $$F_2 F_3\, x = c_1 \quad \equiv \quad F_3\, x = F_2^{-1}\, c_1 = c_2$$

and we find c_2 as the solution of the simple system $F_2\, x = c_1$.

(3) $$F_3\, x = c_2 \quad \equiv \quad x^* = F_3^{-1}\, c_2$$

and we find x^* as the solution of the simple system $F_3\, x = c_2$.

In general, if A is invertible, the solution is determined by solving as many *simple systems* as there are factors of A.

(2.08) Definition (LR factorization, LR factorization with pivoting and QR factorization).

Let $A \in R^{n \times n}$.

An *LR factorization of* A is a pair S,D such that:

- $S \in R^{n \times n}$ is a lower triangular matrix with $s_{kk} = 1$ for $k = 1,\ldots,n$
- $D \in R^{n \times n}$ is an upper triangular matrix
- $S\,D = A$

Note that the left factor S is invertible. Then: A *is invertible if and only if the right factor D is invertible.*

A *LR factorization of* A *with pivoting* is a triplet P,S,D such that:

- $P \in R^{n \times n}$ is a permutation matrix
- the pair S,D is an LR factorizationn of P A

The relationship between A,P,S and D is:

$$P\,A = S\,D \qquad \text{i.e.} \qquad A = P^t\, S\, D$$

Note that both P and the left factor S are invertible. Again: A *is invertible if and only if the right factor* D *is invertible.*

A *QR factorization of* A is a pair U,T such that:

- $U \in R^{n\times n}$ is an orthogonal matrix
- $T \in R^{n\times n}$ is an upper triangular matrix
- $U T = A$

Note that the left factor U is invertible. Again: A *is invertible if and only if the right factor* T *is invertible*.

(2.09) Definition (elementary Gaussian matrix).

Let $A \in R^{n\times n}$, to find an LR factorization with pivoting, the EGP procedure is used, which is based on the *Gaussian elimination* process. To describe the procedure, the notion of an *elementary Gaussian matrix* is needed.

$H \in R^{n\times n}$ is an *elementary Gaussian matrix* if: there exists an index $k \in \{1,...,n-1\}$ and real numbers $\lambda_{k+1},...,\lambda_n$ such that H is obtained from the identity matrix $I \in R^{n\times n}$ by replacing the k-th column e_k (whose components are all equal to zero except the k-th one which is one) with the column:

$$[0 ;...; 0 ; \underset{\substack{\uparrow \\ \text{k-th component}}}{1} ; \lambda_{k+1} ;...; \lambda_n]$$

Examples:

- the identity matrix $I \in R^{n\times n}$ is elementary Gaussian;
- the matrix:

  [1,0,0;
   1,1,0;
  -2,0,1]

  is elementary Gaussian;
- the matrix:

  [1,0,1;
   1,1,0;
  -2,0,1]

  is *not* elementary Gaussian.

(2.10) Properties (of elementary Gaussian matrices).

Let H be an elementary Gaussian matrix. Then:

- H is *lower triangular* with $h_{kk} = 1$ for every k (hence it is *invertible*)
- H^{-1} is obtained from H by *changing the sign* of the elements below the main diagonal

  (for Example:

  H = [1,0,0; H^{-1} = [1,0,0;
  1,1,0;   -1,1,0;
  -2,0,1]   2,0,1])

(2.11) Definition (EGP procedure).

The following EGP procedure operates on a matrix $A \in R^{n\times n}$, and determines a triplet P,S,D which is an LR factorization of A with pivoting.

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(P,S,D) = EGP(A)

  A_1 = A;
  for k = 1,...,n-1 repeat:
    determine appropriately a permutation matrix P_k, an elementary Gaussian
        matrix H_k and set A_{k+1} = H_k P_k A_k;
  D = A_n;
  P = P_{n-1} ... P_1;
  S = P (P_1^t H_1^{-1} … P_{n-1}^t H_{n-1}^{-1})
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The matrices P_k and H_k are determined so that A_n is *upper triangular*.

Observe that:

$$D = A_n = H_{n-1}\ P_{n-1}\ A_{n-1} = \ldots = H_{n-1}\ P_{n-1}\ \ldots\ H_1\ P_1\ A$$

so that:

$$A = (P_1^t\, H_1^{-1}\ \ldots\ P_{n-1}^t\ H_{n-1}^{-1})\ D$$

The matrix $P_1^t\, H_1^{-1}\ \ldots\ P_{n-1}^t\ H_{n-1}^{-1}$ *is not* lower triangular with unit elements on the diagonal but the matrix

$$P\ (P_1^t\, H_1^{-1}\ \ldots\ P_{n-1}^t\ H_{n-1}^{-1})$$

is lower triangular with unit elements on the diagonal. Hence, the pair $S = P\ (P_1^t\, H_1^{-1} \ldots P_{n-1}^t\ H_{n-1}^{-1})$, D is an LR factorization of $P\,A$, as required.

It remains to be clarified how, at each iteration, the matrices P_k and H_k are determined.