Lecture 15 (hrs. 27,28) - October 23, 2025, 8:30 - 10:30 F3

(2) SYSTEMS OF LINEAR EQUATIONS

(2.01) Example.

Examples of contexts in which systems of linear equations must be solved:

- at each iteration of Newton's method for functions from R^n to R^n;
- solving linear resistive electrical networks
- solving linear RLC electrical networks in sinusoidal regime.

(2.02) Problem.

Given an invertible matrix $A \in R^{n\times n}$ and $b \in R^n$, find $x^* \in R^n$ s.t. $A x^* = b$. The column x^* is called a *solution* if the system $A x = b$.

(2.03) Remark.

A matrix $A \in R^{n\times n}$ is invertible if it satisfies one of the following *equivalent* properties:

- there exists a matrix $M \in R^{n\times n}$ such that $AM = MA = I$ (the matrix M is called the *inverse* matrix of A and is denoted by A^{-1})
- $A x = 0 \Leftrightarrow x = 0$ (this property is also expressed by $\ker A = \{ 0 \}$)
- for *each* column $b \neq 0$ in R^n, there is *only one* solution x^* of the system $A x = b$
- $\det A \neq 0$

(2.04) Remark (simple cases).

To decide whether the matrix A of the system is invertible and, if so, to determine the solution of the system $A x = b$ is *simple* when the *structure* of A falls into one of the following cases:

(D) *diagonal* (A is diagonal if $i \neq j \Rightarrow a_{i,j} = 0$)

- It is: $\det A = a_{1,1} \cdots a_{n,n}$, hence: $\det A = 0 \Leftrightarrow$ there exists k s.t. $a_{k,k} = 0$. Then: A *is invertible if and only if for every* k *it is* $a_{k,k} \neq 0$.

- If A is invertible, the components of the solution x^* of the system $A x = b$ are determined by:

$$x_k^* = b_k \ / \ a_{k,k}$$

  The number of operations needed to determine the solution is:

  n *divisions*.

(T) *triangular* (A is *upper* triangular if $i > j \Rightarrow a_{i,j} = 0$; it is *lower* triangular if $i < j \Rightarrow a_{i,j} = 0$)

- Also in this case we have: $\det A = a_{1,1} \cdots a_{n,n}$. Therefore: A *is invertible if and* *only if for each* k *we have* $a_{k,k} \neq 0$.

- If A is an invertible *upper triangular* matrix, the components of the solution x^* of the system $A\,x = b$ are determined by the following *backward substitution* procedure:

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z = BS(T,c)
      if T is not an invertible upper triangular matrix then STOP;
         else
            z_n = c_n / t_{n,n};
            for k = n-1,...,1 repeat
                s = t_{k,k+1} * x_{k+1} + ... + t_{k,n} * x_n;
                x_k = (b_k - s) / t_{k,k};
```

The number of operations needed to determine the solution is:

$$n \textit{ divisions} + \frac{n(n-1)}{2}\ (\textit{multiplications} + \textit{sums})$$

(2.05) Exercise (homework).

Describe the *forward substitution* procedure whose header is:

$$z = FS(T,c)$$

which, given an invertible lower triangular matrix T and a column c, determines the solution of the system $T\,x = c$. Also determine the number of operations necessary to find the solution.