Lecture 14 (hrs. 25,26) - October 22, 2025, 11:30 - 13:30 A13

(1.80) Example.

Let $f:R^2 \to R^2$ be defined by:[1]

$$f(x) = [\ f_1(x_1,x_2)\ ;\ f_2(x_1,x_2)\ ] = [\ x_1^2 - x_2\ ;\ - x_1 + x_2^2\ ]$$

The jacobian matrix of f in x is:

$$J_f(x) = [\ 2\ x_1\ ,\ -1\ ;\ -1\ ,\ 2\ x_2\ ]\ :\ R^2 \to R^{2\times 2}$$

(1.81) Remark.[2]

Given an element x(k) in R^n, Newton's method for the function $f: R^n \to R^n$ determines the element x(k+1) by solving the equation:

$$f(x(k)) + J_f(x(k))\ (x - x(k)) = 0$$

i.e.:

$$J_f(x(k))\ (x - x(k)) = - f(x(k))$$

This last equation is a *system of linear equations*. If the matrix $J_f(x(k))$ is invertible then we obtain:

$$x - x(k) = - J_f(x(k))^{-1}\ f(x(k))$$

Therefore, the element x(k+1) is:

$$x(k+1) = x(k) - J_f(x(k))^{-1}\ f(x(k))$$

(1.82) Example.

Consider the function $f:R^2 \to R^2$ of Example (1.80) and let x(0) = [1 ; -1]. To determine x(1) we need to calculate $J_f(x(0))$, f(x(0)) and then solve the system

$$J_f(x(0))\ z = - f(x(0))$$

It is:

$$J_f(x(0)) = [\ 2\ ,\ -1\ ;\ -1\ ,\ -2\ ] \quad , \quad f(x(0)) = [\ 2\ ;\ 0\ ]$$

Observe that $J_f(x(0))$ is invertible. The solution of the system is:

$$p = [\ - 4/5\ ;\ 2/5\ ]$$

Hence:

$$x(1) = x(0) + p = [\ 1/5\ ;\ -3/5\ ]$$

1 For matrices we will use *Scilab* notation.

2 For sequences of elements in R^n, we will use the notation x(0), x(1), x(2), ...

(1.83) Definition.

Newton's method applied to the function $f:R^n \to R^n$, with invertible Jacobian matrix $J_f(x)$, is the one-point method defined by the function:

$$N(x) = x - J_f(x)^{-1}\, f(x) \;:\; R^n \to R^n$$

(1.84) Theorem (local convergence for one-point methods in R^n).

Let $h: R^n \to R^n$ be sufficiently regular and α be a fixed point of h.

If *all* eigenvalues of $J_h(\alpha)$ have *modulus* < 1 then there exists a real number $\rho > 0$ such that:

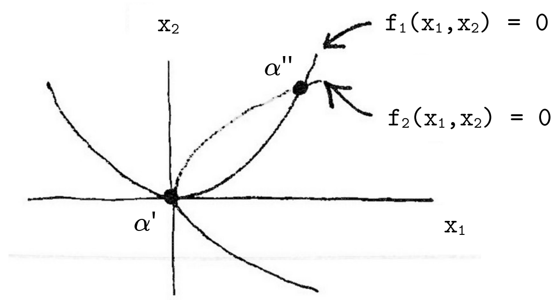
$\|\, x(0) - \alpha \,\| < \rho \quad \Rightarrow$ the sequence x(k) generated by the iterative method defined by h starting from x(0) *converges to* α

This Theorem provides a *sufficient condition for the usability of the method* defined by h to approximate α. For a one-point method in R^n, being usable means that *for every* x(0) *sufficiently close to a fixed point* α *of* h, *the sequence generated by the method defined by* h *starting from* x(0) *converges to* α.

(1.85) Example (part one).

Consider again the function $f:R^2 \to R^2$ from Example (1.80).
The function has *two* zeros:

$$\alpha' = [0 \;;\; 0] \quad , \quad \alpha'' = [1 \;;\; 1]$$



To approximate the two zeros, consider the method defined by the function

$$h(x) = x + f(x) = [\; x_1 + x_2^2 - x_2 \;;\; x_2 - x_1 + x_2^2 \;]$$

It is easily verified that the fixed points of h are all and only the zeros of f.

The jacobian matrix is:

$$J_h(x) = I + J_f(x) = [\; 1 + 2\,x_1 \;,\; -1 \;;\; -1 \;,\; 1 + 2\,x_2 \;]$$

hence:

$$J_h(\alpha') = [\; 1 \;,\; -1 \;;\; -1 \;,\; 1 \;]$$

The eigenvalues are the roots of the characteristic polynomial:

$$p(\lambda) = \det(\; J_h(\alpha') - \lambda I) = (1 - \lambda)^2 - 1 \qquad \text{i.e.} \qquad \lambda_1 = 0 \;,\; \lambda_2 = 2$$

The Local Convergence Theorem does not apply. However, the following holds:

(1.86) Remark.

Under the assumptions of the Local Convergence Theorem: if *at least one* of the eigenvalues of $J_h(\alpha)$ has *modulus* > 1 then the iterative method defined by h *cannot be used* to approximate α.

(1.87) Example.

To justify the previous assertion, consider the following particular case.

Let $h(x) = [\ h_1(x_1)\ ;\ h_2(x_2)\ ]: R^2 \to R^2$ where h_1 and h_2 are regular functions, let α_1 be a fixed point of h_1 and α_2 be a fixed point of h_2. It follows that $\alpha = [\alpha_1\ ;\ \alpha_2]$ is a fixed point of h. The jacobian matrix of h in α is:

$$J_h(\alpha) = [\ h_1'(\alpha_1)\ ,\ 0\ ;\ 0\ ,\ h_2'(\alpha_2)\ ]$$

whose eigenvalues are:

$$\lambda_1 = h_1'(\alpha_1) \quad \text{e} \quad \lambda_2 = h_2'(\alpha_2)$$

Let x(k) be a sequence generated by the method defined by h. Then $x_1(k)$ and $x_2(k)$ are, respectively, a sequence generated by the method defined by h_1 and, respectively, by the method defined by h_2. If, for example, $|\lambda_1| = |h_1'(\alpha_1)| > 1$, for the sequence $x_1(k)$ we have (Remark (1.61) of Lecture 10): either $x_1(k) = \alpha_1$ for a finite value of k or $x_1(k)$ does not converge to α_1. As already observed, the possibility of the first condition being met is very remote. Therefore, the sequence is expected not to be convergent. If in this situation the iterative method defined by h were usable to approximate α, then for any x(0) sufficiently close to α the sequence x(k) would converge to the fixed point of h. It would follow that for any $x_1(0)$ sufficiently close to α_1 the sequence $x_1(k)$ would converge to the fixed point of h_1. But this, as observed above, is not possible.

(1.88) Example (part two).

From the final result of part one of the example it can be deduced that the method defined by h *cannot be used* to approximate α'.

Consider now α''. It is:

$$J_h(\alpha'') = [\ 3\ ,\ -1\ ;\ -1\ ,\ 3\ ]$$

hence:

$$p(\lambda) = \det(\ J_h(\alpha'') - \lambda I) = (3 - \lambda)^2 - 1 \quad \text{i.e.} \quad \lambda_1 = 2\ ,\ \lambda_2 = 4$$

and the method defined by h *is not usable* to approximate α'' too.

(1.89) Exercise (homework).

Let f be the function of Example (1.85). Determine the function $N: R^2 \to R^2$ that defines Newton’s method applied to f and verify (with a lot of patience) that we have: $J_N(\alpha') = 0$ and $J_N(\alpha'') = 0$.

(1.90) Remark (usability of Newton’s method).

What was shown in the previous exercise is generally valid. In fact, we have:

If f has continuous second derivatives, J_f is non-singular, and α is a zero of f, then $J_N(\alpha) = 0$ and Newton's method *can be used* to approximate α. Furthermore, similarly to what happens in the case of functions of one variable, the *order of convergence* to α of Newton's method is *at least two*.