Lecture 13 (hrs. 23,24) - October 21, 2025, 16:30 - 18:30 F3

(1.73) Remark (halt condition, part two).

Both the halt conditions considered in Remark (1.72) of Lecture 12 present the problem that, in some cases, x_k is an insufficiently good approximation of α. This arises from the fact that, in the halt condition, estimating the absolute error committed by approximating α with the last element of the sequence calculated ($|x_k - \alpha|$) using the chosen quantity ($|x_{k+1} - x_k|$ in one case, $|f(x_k)|$ in the other), a relative error is committed that *does not tend to zero* as $k \to \infty$.

The two conditions can be modified to obtain better estimates. Using the same context as the two previous conditions:

(1-bis) Given a positive real number E (the maximum error required by the user) and inserting E and the derivative h' between the input variables of the procedure:

<u>if</u> $|x_{k+1} - x_k| \;/\; |1 - h'(x_k)| < E$ <u>then</u> STOP

The condition is *computable* and *effective*.

To understand how good x_k is as an approximation of α when the condition is verified, note that, proceeding as in (1) of Remark (1.72):

$$\left|\frac{x_{k+1}-x_k}{1-h'(x_k)}\right| = \left|\frac{1-h'(t_k)}{1-h'(x_k)}\right| \; |x_k - \alpha| = (1 + \varepsilon_k) \; |x_k - \alpha|$$

where

$$\varepsilon_k = \frac{h'(x_k)-h'(t_k)}{1-h'(x_k)}$$

In this case, when $k \to \infty$ we have $x_k \to \alpha$, $t_k \to \alpha$ and hence $\varepsilon_k \to 0$.

(2-bis) Given a positive real number E (the maximum error required by the user) and insert E, f and f' among the input variables of the procedure:

<u>if</u> $|f(x_k)|/|f'(x_k)| < E$ <u>then</u> STOP

The condition is *computable* and *effective*.

To understand how good x_k is as an approximation of α when the condition is verified, note that, proceeding as in (2) of Remark (1.72):

$$\left|\frac{f(x_k)}{f'(x_k)}\right| = \left|\frac{f'(t_k)}{f'(x_k)}\right| \; |x_k - \alpha| = (1 + \varepsilon_k) \; |x_k - \alpha|$$

where

$$\varepsilon_k = \left|\frac{f'(t_k)}{f'(x_k)}\right| - 1$$

In this case too, when $k \to \infty$ we have $x_k \to \alpha$, $t_k \to \alpha$ and hence $\varepsilon_k \to 0$.

(1.74) Remark (one-point methods in $F(\beta,m)$).

Let:

- $h:[a,b] \to R$ and γ in $[a,b]$ verify the hypotheses of convergence Theorem (1.59) of Lecture 9
- $\varphi:[a,b] \to F(\beta,m)$ the algorithm used to approximate the values of h, s.t.:

$$\text{for every } \theta \text{ in } [a,b] \cap F(\beta,m) \ , \ |\varphi(\theta) - h(\theta)| \leqslant d_\varphi$$

Then, let x_k be the sequence generated by the method defined by h starting from γ, convergent to α by hypothesis, and let ξ_k be the sequence defined by $\xi_0 = \gamma$, $\xi_{k+1} = \varphi(\xi_k)$. Suppose that for each k it is ξ_k in $[a,b]$.

We have:

(1.75) Theorem (stability of one-point methods, part I).

Let $\delta > 0$. If MetodoUnPunto(h,a,b,δ) executed in $F(\beta,m)$ defines ξ in $F(\beta,m)$ such that

$$|\xi_{k+1} \ominus \xi_k| < rd(\delta)$$

then ξ is a fixed point of a function $h^*:[a,b] \to R$ such that:

$$\text{for every } x \text{ in } [a,b] \ , \ |h^*(x) - h(x)| \leqslant d_\varphi + \delta$$

Informally: if d_φ is ‘small’, the procedure returns a fixed point of a function h^* ‘close’ to h.

(1.76) Theorem (stability of one-point methods, part II).

Moreover, let $f:[a,b] \to R$ be a regular function such that $f(\alpha) = 0$, and $\psi:[a,b] \to F(\beta,m)$ be the algorithm used to approximate the values of f such that:

$$\text{for every } \theta \text{ in } [a,b] \cap F(\beta,m) \ , \ |\psi(\theta) - f(\theta)| \leqslant d_\psi$$

Let $\delta > 0$. If MetodoUnPunto(h,a,b,f,δ) executed in $F(\beta,m)$ defines ξ in $F(\beta,m)$ s.t.

$$|\psi(\xi_k)| < rd(\delta)$$

then ξ is a zero of a function $f^*:[a,b] \to R$ s.t.:

$$\text{for every } x \text{ in } [a,b] \ , \ |f^*(x) - f(x)| \leqslant d_\psi + \delta$$

Informally: if d_ψ is ‘small’, the procedure returns a zero of a function f^* ‘close’ to f.

(1.77) Remark (*effectiveness of the halt conditions in F(β,m)).

The two previous theorems state that if in F(β,m) the procedure defines ξ then... This suggests that the procedure *might not define* ξ. The assumption is correct: as we already know, in F(β,m) the halt conditions *may prove ineffective*.

Example.

Let [a,b] not contain 0. Then A = [a,b] ∩ F(β,m) contains a finite number of elements. Let $\Delta > 0$ be the minimum distance between two consecutive elements of A. If φ has no fixed points in [a,b], then we have:

$$|\xi_{k+1} - \xi_k| \geqslant \Delta \qquad \text{hence} \qquad |\xi_{k+1} \ominus \xi_k| \geqslant \Delta$$

If the user chooses $\delta < \Delta$, the condition $|\xi_{k+1} \ominus \xi_k| < \mathrm{rd}(\delta)$ *cannot* be verified.

In the other case, if ψ has no zeros in [a,b], let $\Gamma > 0$ be the minimum value of ψ in A, we have:

$$|\psi(\xi_k)| \geqslant \Gamma$$

If the user chooses $\delta < \Gamma$, the condition $|\psi(\xi_k)| < \mathrm{rd}(\delta)$ cannot be verified.

(1.78) Example.

Let $f(x) = (x - 2)^2$. The function has only one zero, $\alpha = 2$ and $f'(\alpha) = 0$. Choosing $x_0 > 2$, for the sequence generated by Newton's method applied to f we have:

$$x_{k+1} = (x_k + 2) / 2$$

hence:

$$x_k - 2 = (1/2)^k (x_0 - 2)$$

The sequence converges to α but is an *exponential sequence*. In this case we have:

$$h_N(x) = (x + 2) / 2$$

hence $h'(\alpha) = 1/2 \neq 0$: Newton's method applied to f has an *order of convergence to α equal to one*.

(1.3) NEWTON'S METHOD FOR FUNCTIONS FROM R^n TO R^n

(1.79) Remark.

If $f:R \to R$ is a regular function, each iteration of Newton's method constructs, starting from a known value x_k, the real number x_{k+1} by determining the zero (if it exists) of the affine function (see Remark (1.67) in Lecture 11):

$$A_k(x) = f(x_k) + f'(x_k)\ (x - x_k)$$

The function $A_k:R \rightarrow R$ is the Taylor expansion of $f(x)$ of order one in x_k (graphically: the line with equation $y = A_k(x)$ is the tangent to the graph of $f(x)$ in x_k).

The idea of Newton's method in the case where $f: R^n \rightarrow R^n$ is a regular function is the same: at each iteration, starting from a known value $x_k \in R^n$, we construct the zero (if it exists) of the Taylor expansion of $f(x)$ of order one in x_k:

$$A_k(x) = f(x_k) + J_f(x_k)\ (x - x_k)$$

where $J_f(x) \in R^{n \times n}$ is the *jacobian matrix* of f in x, i.e. the matrix whose i,j element is:

$$\frac{\partial f_i}{\partial x_j}(x)$$