Lecture 12 (hrs. 21,22) - October 16, 2025, 8:30 - 10:30 F3

(1.71) Problem.

Let t be a positive real number, n an integer $\geqslant 2$ and $f(x) = x^n - t$. The function f has only one zero, the nth root of t: $t^{1/n}$.

Decide whether Newton's method is applicable to approximate the zero and, if so, determine x_0 such that the method generates a sequence convergent to zero.

(1.72) Remark (halt condition).

The halt condition presented for the bisection method *cannot* be used for one-point methods: these latter methods, unlike the bisection method, do not generate a sequence of intervals of measure tending to zero, each containing a zero of the function. Different conditions are therefore required. Let's discuss the two most commonly used, both of which are of *absolute* type.

Let f be the function of which we wish to approximate a zero, $h:[a,b] \to R$ and γ verify the hypotheses of the Convergence Theorem, α be the fixed point of h (and zero of f) in [a,b] and x_k be the sequence generated by the method defined by h starting from γ. The sequence x_k converges to α.

(1) Given a positive real number E (the maximum error required by the user) and added E to the input variables of the procedure:

$$\underline{\text{if}}\ |x_{k+1} - x_k| < E\ \underline{\text{then}}\ \text{STOP}$$

The condition is *computable*: at each iteration the procedure knows x_k, determines $x_{k+1} = h(x_k)$ and verifies if the condition is satisfied.

The condition is *effective*: both the sequence x_k and the sequence $x_{k+1} = h(x_k)$ converge to α (the function h is continuous and α is a fixed point of h), so the difference $x_{k+1} - x_k$ tends to zero. The condition is certainly satisfied after a finite number of iterations.

To understand how good x_k is as an approximation to α when the condition is satisfied, note that:

$$|x_{k+1} - x_k| = |h(x_k) - x_k| = |(h(x_k) - \alpha) + (\alpha - x_k)| = |(h(x_k) - h(\alpha)) + (\alpha - x_k)|$$

Using Lagrange's Theorem:

$$h(x_k) - h(\alpha) = h'(t_k)(x_k - \alpha) \quad \text{with} \quad t \text{ between } x_k \text{ and } \alpha$$

hence:

$$|x_{k+1} - x_k| = |h'(t_k)(x_k - \alpha) + (\alpha - x_k)| = |h'(t_k) - 1|\ |x_k - \alpha| = |1 - h'(t_k)|\ |x_k - \alpha|$$

The accuracy of x_k as an approximation of α *depends on* the value of $h'(t_k)$. Precisely:

- if $h'(t_k) \approx 0$ we have $|x_{k+1} - x_k| \approx |x_k - \alpha|$ and the halt condition interrupts the construction of the sequence *as soon as* the approximation is accurate (note that *if* f *is sufficiently regular and* $f'(\alpha) \neq 0$ *Newton's method has* $h'(\alpha) = 0$ *and for a sufficiently large value of k it is* $h'(t_k) \approx 0$);

- if $h'(t_k) \approx 1$ we have $1 - h'(t_k) \approx 0$ and the halt condition interrupts the construction of the sequence *before* the approximation is accurate;
- if $h'(t_k) < 0$ it is $|1 - h'(t_k)| > 1$ hence $|x_{k+1} - x_k| < E \Rightarrow |x_k - \alpha| < E$ (but the halt condition could interrupt the construction of the sequence *too late*: the approximation could be good already a few iterations before).

  Example: Let $h(x) = \alpha + A(x - \alpha)$ and $h'(x) = A$. For every k it is: $x_k - \alpha = A^k(x_0 - \alpha)$.

  If $A = 0.9$ (≈ 1) *and* k is such that $|x_{k+1} - x_k| = 0.99$ E (halt condition satisfied), *then* $|x_k - \alpha| = (0.99 / 0.1)$ E $= 9.9$ E $>$ E and the accuracy of the approximation does not verify the user's request.

  If $A = -0.9$ *and* k is such that $|x_{k+1} - x_k| = 0.99$ E (halt condition satisfied), *then* $|x_k - \alpha| = E / 1.9 \approx 0.5$ E $<$ E and the accuracy of the approximation verifies the user's request. *But*: 6 iterations before we already had $|x_{k-6} - \alpha| = |x_k - \alpha| / |A|^6 = E / (1.9\ 0.9^6) = E / 1.009... < E$, i.e. *6 iterations before* the approximation already satisfied the user's request.

(2) Given a positive real number E (the maximum error required by the user) and added both E and f to the input variables of the procedure:

$$\underline{\text{if}}\ |f(x_k)| < E\ \underline{\text{then}}\ \text{STOP}$$

The condition is *computable*: at each iteration the procedure knows xk, determines f(xk) and verifies if the condition is satisfied.

The condition is *effective*: the sequence x_k converges to α and the sequence $f(x_k)$ converges to $f(\alpha) = 0$ (the function f is continuous and α is a zero of f). The condition is therefore certainly satisfied after a finite number of iterations.

To understand how good an approximation x_k is to α when the condition is satisfied, suppose f is regular and observe that:

$$f(x_k) = f(x_k) - f(\alpha)$$

Using Lagrange's Theorem:

$$f(x_k) - f(\alpha) = f'(t_k)(x_k - \alpha) \quad \text{with} \quad t_k \text{ between } x_k \text{ and } \alpha$$

hence:

$$|f(x_k)| = |f'(t_k)|\ |x_k - \alpha|$$

The accuracy of x_k as an approximation of α *depends on* the value of $|f'(t_k)|$. Precisely:

- if $|f'(t_k)| \approx 1$ it is $|f(x_k)| \approx |x_k - \alpha|$ and the halt condition interrupts the construction of the sequence *as soon as* the approximation is accurate;
- if $|f'(t_k)| \approx 0$ the halt condition interrupts the construction of the sequence *before* the approximation is accurate;
- if $|f'(t_k)| > 1$ it is $|f(x_k)| < E \Rightarrow |x_k - \alpha| < E / |f'(t_k)| < E$ (but the halt condition could interrupt the construction of the sequence *too late*: the approximation could be good already a few iterations before).