Lecture 11 (hrs. 19,20) - October 15, 2025, 11:30 - 13:30 A13

(1.2) NEWTON'S METHOD

(1.64) Definition (Newton's method).

Let $f:[a,b] \to R$ be a function with first derivative such that $f'(x) \neq 0$ for all x in [a,b].

Newton's method applied to the function f is the one-point method defined by the function $h_N:[a,b]\to R$ such that:

$$h_N(x) = x - (f'(x))^{-1} f(x) = x - \frac{f(x)}{f'(x)}$$

Note that the *fixed points of* h_N are all and only the *zeros of* f.

(1.65) Remark (usability of Newton's method).

Let $f:[a,b] \to R$ be a function with continuous *second* derivative and with $f'(x) \neq 0$ for all x in [a,b]. Then let α be a zero of f in [a,b]. We have:

$$h_N'(x) = 1 - \frac{(f'(x))^2 - f''(x)f(x)}{(f'(x))^2} = \frac{f''(x)f(x)}{(f'(x))^2}$$

The function h_N' is continuous and, since $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, we have

$$h_N'(\alpha) = 0$$

By Theorem (1.59) of Lecture 10, Newton's method *can be used* to approximate α.

(1.66) Remark (usability condition of Newton's method).

Let $f:[a,b] \to R$ be a function with a continuous second derivative and α be a zero of f in [a,b]. A *sufficient* condition for Newton's method applied to f to be *usable* to approximate α is:

$$f'(\alpha) \neq 0$$

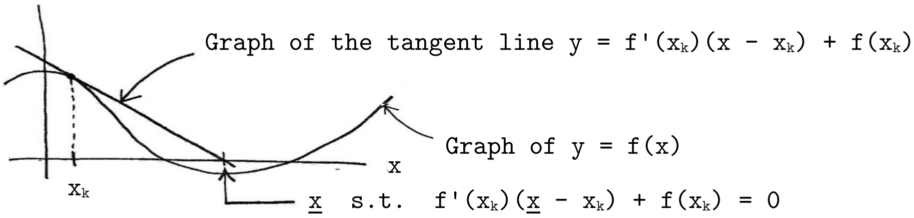
(1.67) Remark (graphical construction for Newton's method).

Let $f:[a,b] \to R$ be a function with first derivative and let x_k be a real number such that $f'(x_k) \neq 0$. Draw on the same Cartesian plane the graph of the function f and the graph of the tangent line to the graph of f at x_k (see figure). Since $f'(x_k) \neq 0$, the tangent line is not horizontal and therefore intersects the x-axis at the point $\underline{x}$ such that:

$$f'(x_k)(\underline{x} - x_k) + f(x_k) = 0$$

i.e. at the point

$$\underline{x} = x_k - \frac{f(x_k)}{f'(x_k)} = h_N(x_k)$$



(1.68) Remark (how to choose the starting point in Newton's method).

Let f:[a,b] → R with continuous second derivative be such that:

(1) there exists α zero of f in [a,b]
(2) for every $x \in [a,b]$ we have $f'(x) \neq 0$ (hence α is *the unique* zero of f in [a,b])
(3) $f''(x) \neq 0$ (f is *convex* on [a,b])

Then: starting from γ = *the endpoint of* [a,b] *where* f *and* f'' *have the same sign*, Newton's method generates a sequence in [a,b] *convergent to* α *and monotone*.

(Proof. Using the hypotheses and graphical reasoning, we show that the sequence generated from γ is *monotone and bounded*, and therefore convergent. The limit can only be a fixed point of h_N in [a,b], hence the limit should be α.)

(1.69) Remark.

Let f:[a,b] → R be a function with a continuous second derivative and α be a zero of f in [a,b]. If $f'(\alpha) \neq 0$ (thus Newton's method applied to f can be used to approximate α), then there exists an interval I that satisfies the hypotheses of Remark (1.68) if and only if $f''(\alpha) \neq 0$.

(1.70) Remark (order of convergence to a fixed point of a method).

Let h:[a,b] → R, α be a fixed point of h and x_k be a sequence convergent to α and generated by the method defined by h.

(1) Let h have a continuous first derivative such that $0 < |h'(\alpha)| < 1$. Then:

- Let $d > 0$ such that $h'(x) \neq 0$ for every $x \in I(\alpha, d)$. Let λ_d and L_d be the minimum and maximum of $|h'(x)|$ on $I(\alpha, d)$, respectively, and $y_{n,d}$ be the sequence consisting of the elements of x_k in $I(\alpha, d)$. For every x in $I(\alpha, d)$, we have:

$$\lambda_d \leqslant |h'(x)| \leqslant L_d$$

- Then, for every n we have:

$$\lambda_d^n \, |y_{0,d} - \alpha| \leqslant |y_{n,d} - \alpha| \leqslant L_d^n \, |y_{0,d} - \alpha|$$

i.e.:

the sequence $y_{n,d} - \alpha$ converges to zero *faster* than the sequence $L_d^n \, |y_{0,d} - \alpha|$ but *slower* than the sequence $\lambda_d^n \, |y_{0,d} - \alpha|$

Given a tiny real number d it is $\lambda_d \approx L_d \approx |h'(\alpha)|$. Hence:

$$|y_{n,d} - \alpha| \approx |h'(\alpha)|^n \, |y_{0,d} - \alpha|$$

This property of the sequence x_k is expressed by saying that 'x_k *converges exponentially* to α'.

(2) Let $h(x) = \alpha + A(x - \alpha)^2$ with $A \neq 0$. Then: α ia a fixed point of h and $h'(\alpha) = 0$. Furthermore, given a real number x_0, for every k we have:

$$x_k - \alpha = A^{-1} \left(A(x_0 - \alpha) \right)^{2^k}$$

If $|A(x_0 - \alpha)| < 1$, the sequence x_k converges to α and, for every t in (0,1) we have

$$\frac{|x_k - \alpha|}{t^k} \longrightarrow 0 \quad \text{when } k \to \infty$$

that is: the sequence $x_k - \alpha$ tends to zero *more rapidly* than *any* exponential sequence.

In general, if h has a continuous second derivative and $h'(\alpha) = 0$, the sequence x_k tends to α *more rapidly* than *any* exponential sequence.

When the conditions 'h with continuous h' and $0 < |h'(\alpha)| < 1$' hold, we say that *the order of convergence to α of the method defined by* h *is* <u>*one*</u>. Analogously, when the conditions 'h with continuous h'', $h'(\alpha) = 0$ and $h^{(2)}(\alpha) \neq 0$' hold, we say that *the order of convergence to α of the method defined by* h *is* <u>*two*</u>. In general:

the order of convergence to α of the method defined by h *is p*

means

the function $h^{(p)}(x)$ is continuous, $h^{(m)}(\alpha) = 0$ for $m = 1,\ldots,p - 1$ and $h^{(p)}(\alpha) \neq 0$

The higher the order of convergence to α of the method, the more rapidly the sequences generated by the method converge to α.