Lecture 10 (hrs. 17,18) - October 14, 2025, 16:30 - 18:30 F3

(1.58) Remark.

The use of the Convergence Theorem (Theorem (1.57) of Lesson 9) requires the verification of hypotheses (1) - (3). For hypotheses (1) and (2) it is necessary to decide whether there exists, and possibly determine, an interval [a,b] which contains only one fixed point of h and in all points x of which $|h'(x)| \leqslant L$ with $0 \leqslant L < 1$. Once an interval [a,b] with the required properties has been determined, it is necessary to decide whether hypothesis (3) is verified, that is, whether starting from γ the method defined by h generates a sequence in [a,b].

The following theorem and remark provide concrete criteria for testing hypotheses (1) - (3).

(1.59) Theorem (usability of the method defined by h).

Let $h:[a,b] \to R$ be a function with continuous first derivative and α be a fixed point of h in [a,b]. A *necessary and sufficient condition* to the existence of an interval $I \subset [a,b]$ containing α and at all whose points x it is $|h'(x)| \leqslant L$ with $0 \leqslant L < 1$ is:

$$|h'(\alpha)| < 1$$

Proof.

The condition is *necessary*: if there exists an interval $I \subset [a,b]$ containing α and at all whose points x it is $|h'(x)| \leqslant L$ with $0 \leqslant L < 1$, certainly we have $|h'(\alpha)| < 1$.

The condition is *sufficient*: if $|h'(\alpha)| < 1$, by the continuity of the function h' there exists a real number L with $0 \leqslant L < 1$ and an interval $I \subset [a,b]$ containing α and at all whose points x it is $|h'(x)| \leqslant L$.

(1.60) Remark (how to choose the initial point).

Let $h:[a,b] \to R$ be a function with continuous first derivative that satisfies hypotheses (1) and (2) of the Convergence Theorem and let α be the unique fixed point of h in [a,b]. Then:

using γ = *the one of the endpoints of* [a,b] *closest to* α as initial point, the method defined by h generates a sequence in [a,b], which therefore converges to α.

Proof.

Let $x_0 = \gamma$, and let $d = |x_0 - \alpha|$. Denoted by $I(\alpha,d)$ the neighborhood of center α and radius d, we have $I(\alpha,d) \subset [a,b]$. As shown in point (B) of the Proof of the Convergence Theorem, we have $|x_1 - \alpha| < |x_0 - \alpha| = d$, so $x_1 \in I(\alpha,d)$. Similarly, it can be shown that for every k we have $x_k \in I(\alpha,d) \subset [a,b]$.

(1.61) Remark.

Let $h:[a,b] \to R$ be a function with continuous first derivative, α a fixed point of h and x_k a sequence generated by the method defined by h. If $|h'(\alpha)| > 1$ then one and only one of the following statements hold:

- there exists $\overline{k}$ s.t. for every $k \geqslant \overline{k}$ it is $x_k = \alpha$
- $x_k \nrightarrow \alpha$

(<u>Proof</u> only in a particular case. Let $h(x) = A(x - \alpha) + \alpha$ with $A > 1$. We have: α is the unique fixed point of h, $h'(x) = A$ and

$$x_k - \alpha = A^k(x_0 - \alpha)$$

Then: if $x_0 \neq \alpha$, for every $M > 0$ there exists n s.t. $k \geqslant n \Rightarrow |x_k - \alpha| \geqslant M$. Hence for every $x_0 \neq \alpha$ it is $x_k \nrightarrow \alpha$.)

The possibility of being able to determine an initial point s.t. $x_k = \alpha$ after a *finite* number of terms is extremely remote. For this reason, if $|h'(\alpha)| > 1$ the method defined by h is declared *unusable* to approximate α.

It remains to be clarified what happens if $|h'(\alpha)| = 1$. We will see that even in this case the method defined by h is declared *unusable* to approximate α.

Finally, note that the condition $|h'(\alpha)| < 1$, necessary and sufficient for the usability of the method to approximate the fixed point α, can be verified *graphically* by comparing the slope ($h'(\alpha)$) of the tangent line to the graph of $y = h(x)$ at $x = \alpha$ with that (1) of the graph line of $y = x$ and with that (-1) of the line $y = \alpha - x$.

(1.62) <u>Exercize</u>.

For any $x > 0$, let $f(x) = x + \log(x)$. We want (i) to know whether f has any zeros, and if so: (ii) to separate the zeros, and finally, (iii) to decide whether each of the methods defined by

$$h_1(x) = -\log(x) \quad ; \quad h_2(x) = \exp(-x) \quad ; \quad h_3(x) = (\exp(-x) + x)/2$$

is usable to approximate the zeros of f.

<u>Solution</u>.

(i) The function f(x) is continuous, $f(x) \to -\infty$ when $x \to 0$ and $f(x) \to +\infty$ when $x \to +\infty$. It follows that f has *at least one* zero. The function f(x) is also differentiable and for every $x > 0$ we have $f'(x) \neq 0$. Then f has *at most one* zero. Therefore f has a zero, α.[1]

(ii) We have: $f(1) = 1$, therefore $\alpha \in [0,1]$, that is, the interval [0,1] *separates* the zero of f.

(iii) Consider the function $h_1(x)$. It is easily verified that the zeros of f are all and only the fixed points of h_1. Furthermore, h_1 is differentiable and for every $x > 0$ we have $h_1'(x) = 1/x$. Since $\alpha \in (0,1)$ we certainly have $|h_1'(\alpha)| > 1$. By Remark (1.61) the method defined by h_1 *cannot be used* to approximate α.

1 Let $f:[a,b] \to R$ be a sufficiently regular function. If for every x in [a,b] we have

$$f^{(k)}(x) \neq 0$$

then f has *at most* k *distinct zeros* in the interval [a,b].

Consider the function $h_2(x)$. It is easily verified that the zeros of f are all and only the fixed points of h_2. Furthermore, h_2 is differentiable and for every x we have $|h_2'(x)| = \exp(-x)$. Since $\alpha \in (0,1)$ we certainly have $|h_2'(\alpha)| < 1$ and, by Theorem (1.59), the method defined by h_2 *can be used* to approximate α. According to Remark (1.60), to determine an initial point from which the method defines a sequence convergent to α it is sufficient to determine a closed interval I that verifies hypotheses (1) and (2) of the Convergence Theorem. The interval [0,1] *is not suitable* because hypothesis (2) is not verified: for every x in (0,1] we have $0 \leqslant |h_2'(x)| = \exp(-x) < 1$ but $|h_2'(0)| = 1$. Then, an interval that also verifies hypothesis (2) is [t,1] with $t \in (0,\alpha)$. To determine t we use the Bolzano's Theorem. Since $f(1/2) < 0$, we set t = 1/2 and I = [1/2, 1]. At this point it is sufficient to decide which of the two extremes of I is closer to zero. We use the Bolzano's Theorem again. Since $f(3/4) > 0$, we choose $x_0 = 1/2$.

Note that, in this case, for every x in I = [1/2, 1] the first derivative of the function defining the method is *negative*. Since, reviewing the Proof of the assertion (B) of the Convergence Theorem, for every k we have:

$$x_k - \alpha = h'(t_{k-1})(x_{k-1} - \alpha)$$

for some real number t_{k-1} in I, then for all k $h'(t_{k-1}) < 0$ and the differences $x_k - \alpha$ and $x_{k-1} - \alpha$ have *opposite signs*. It follows that the elements of the sequence are found, alternatively, to the right and to the left of α: the sequence 'oscillates' around zero. Anyway, the sequence of *distances* $|x_k - \alpha|$ is *monotonically decreasing* as shown in the Proof of the Convergence Theorem.

Finally, consider the function $h_3(x)$. It is easy to verify that the zeros of f are all and only the fixed points of h_3. Furthermore, h_3 is differentiable and for each x we have:

$$|h_3'(x)| = (1 - \exp(-x))/2$$

Since $\alpha \in (1/2,1)$ we certainly have $|h_3'(\alpha)| < 1$ and, by Theorem (1.59), the method defined by h_3 *can be used* to approximate α. According to Remark (1.60), to determine an initial point from which the method defines a sequence convergent to α it is sufficient to determine a closed interval I that verifies hypotheses (1) and (2) of the Convergence Theorem. The interval I = [1/2,1] is fine, in fact for every x in I we have $0 \leqslant |h_3'(x)| < 1$. At this point it is sufficient to decide which of the two extremes of I is closer to the zero α. Proceeding as in the previous case, we choose $x_0 = 1/2$.

Note that, in this case, for every x in I = [1/2, 1] the first derivative of the function defining the method is *positive*. Reasoning as in the previous case, the differences $x_k - \alpha$ and $x_{k-1} - \alpha$ have the *same sign*. It follows that the elements of the sequence are all on the same side with respect to α. Furthermore, also in this case, the sequence of *distances* $|x_k - \alpha|$ is *monotonically decreasing*, and therefore the sequence x_k is *monotone* (*increasing* if x_0 is to the left of α, *decreasing* in the opposite case). Finally, note that since for every x in I = [1/2, 1] the first derivative of the function defining the method is positive, from the Proof of the criterion for choosing the initial point (Remark (1.60)) we deduce that *for every* x_0 *in* I the sequence x_k converges to α.

(1.63) <u>Homework</u>.

For every $x \in R$ let: $h(x) = 2\ \mathrm{arctg}(x)$.

(1) Determine the number of fixed points of h and separate them.
(2) For each of the fixed points, decide whether the iterative method defined by h can be used for the approximation and, if so, indicate an initial point from which the generated sequence converges to the fixed point under consideration.
(3) Answer the previous questions using graphical methods, with the help of *Scilab*.