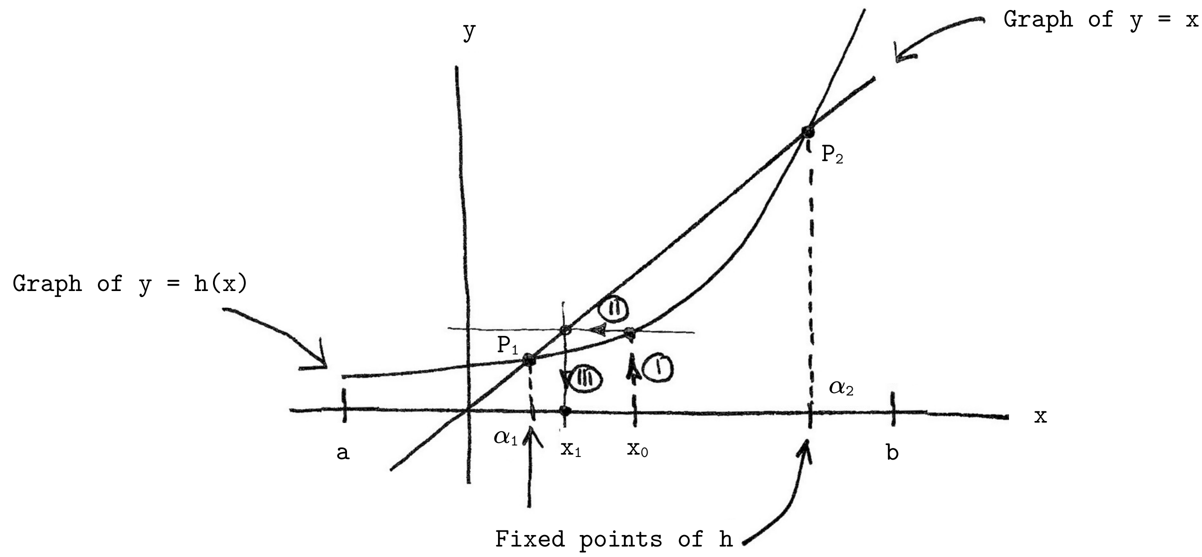
Lecture 9 (hrs 15,16) - October 9, 2025, 8:30 - 10:30 F3

(1.56) Remark (graphical constructions).



Represent on the same Cartesian plane the portions of the graph of the function y = h(x) which defines the one-point method to be examined and of the straight line graph of the function y = x, on an interval [a,b].

The fixed points of h are the abscissas (α_1 e α_2) of the points P_1 and P_2 common to the two graphs.

Given the point on the x-axis representing x_0, we can construct the point on the same axis representing x_1 in three steps: (I) determine the point $(x_0,h(x_0)) = (x_0,x_1)$ intersection of the graph of y = h(x) and the vertical line through $(x_0,0)$; (II) determine the point $(h(x_0),h(x_0)) = (x_1,x_1)$ intersection of the graph of y = x and the horizontal line through the point $(x_0,h(x_0))$ determined in the previous step; (III) determine the point $(h(x_0),0) = (x_1,0)$ intersection of the x-axis and the vertical line passing through (x_1,x_1).

(1.57) Theorem (convergence).

Let h:[a,b] → R a function with continuous first derivative and γ a point in [a,b] such that:

(1) there exists a fixed point α of h in [a,b];
(2) there exists a real number $L \in [0,1)$ such that: for every $x \in [a,b]$ it is $|h'(x)| \leqslant L$;
(3) the procedure MetodoUnPunto(h,a,b,γ) defines a sequence x_k.[1]

Then we have:

(A) α is *the unique* fixed point of h in [a,b];
(B) the sequence x_k *converges* to α.

1 That is, for every k it is: if $x_k \in [a,b]$ then $x_{k+1} \in [a,b]$.

(1.58) <u>Proof</u> (of Theorem (1.57)).

(A) By contradiction. If β is another fixed point of h in [a,b] we have (first using the definition of fixed point and then Lagrange's Theorem):

$$\beta - \alpha = h(\beta) - h(\alpha) = h'(t)(\beta - \alpha) \quad , \quad \text{where } t \text{ is a real number between } \alpha \text{ and } \beta$$

Finally, recalling that $\beta - \alpha \neq 0$, we get:

$$h'(t) = 1 \tag{\#}$$

But, since both α and β are points in [a,b], t is too. Then, by hypothesis (2), equality (#) is absurd.

Note that to prove (A) *only* hypotheses (1) and (2) were used.

(B) It must be shown that the sequence x_k tends to α, i.e. that the sequence $x_k - \alpha$ tends to zero. We have (using Lagrange's Theorem for the second equality):

$$x_k - \alpha = h(x_{k-1}) - h(\alpha) = h'(t_{k-1})(x_{k-1} - \alpha) \quad \text{where} \quad t_{k-1} \text{ between } x_{k-1} \text{ and } \alpha$$

Taking the absolute values we have (the inequality is obtained by using hypothesis (2)):

$$|x_k - \alpha| = |h'(t_{k-1})| \; |x_{k-1} - \alpha| \leqslant L \; |x_{k-1} - \alpha|$$

If k - 1 > 0 we can repeat the reasoning starting from $x_{k-1} - \alpha$ to obtain:

$$|x_{k-1} - \alpha| = |h'(t_{k-2})| \; |x_{k-2} - \alpha| \leqslant L \; |x_{k-2} - \alpha|$$

and, replacing in the previous one:

$$|x_k - \alpha| \leqslant L^2 \; |x_{k-2} - \alpha|$$

Iterating backwards to the first element of the sequence we obtain:

$$|x_k - \alpha| \leqslant L^k \; |x_0 - \alpha|$$

Remembering that $0 \leqslant L < 1$ we obtain the desired result:

$$\lim_{k \to \infty} |x_k - \alpha| = 0$$