Lecture 8 (hrs. 13,14) - October 8, 2025, 11:30 - 13:30 A13

(1.48) Example (numerical approximation of the derivative).

Suppose we know, at instants t_1 and t_2, the positions x_1 and x_2 of a point moving on a straight line. The quantity:

$$\bar{v} = (x_2 - x_1) / (t_2 - t_1)$$

is the average velocity of the point between the two instants. If the quantities x_1 and x_2 are known only with relative error ε_1 and ε_2, for example because they were obtained through measurements, we can only obtain an approximation of $\bar{v}$:

$$w = \frac{(1 + \varepsilon_2)x_2 - (1 + \varepsilon_1)x_1}{t_2 - t_1}$$

The relative error committed by approximating $\bar{v}$ with w is:

$$\frac{w - \bar{v}}{\bar{v}} = \frac{x_2}{x_2 - x_1}\,\varepsilon_2 + \frac{x_1}{x_2 - x_1}\,\varepsilon_1$$

In the case where the difference $x_2 - x_1$ is small (for example when $\bar{v}$ is used as an estimate of the instantaneous velocity of a moving point with high velocity), as shown in the previous Remark, the calculation is *ill-conditioned* and the error committed by approximating $\bar{v}$ with w will be much greater than the single errors ε_1 and ε_2.

(1.49) Exercize.

The statement:

$$\text{(A)} \qquad x = a + \delta \quad \text{with} \quad |\delta| \leqslant d$$

is *equivalent* to:

$$\text{(B)} \qquad x \in [a - d\ ,\ a + d]$$

We want to determine y and E so that the statement:

$$(*) \qquad x = (1 + \varepsilon)y \quad \text{with} \quad |\varepsilon| \leqslant E$$

be equivalent to (A) and (B).

The statement (*) is equivalent to:

$$x \in [(1 - E)y\ ,\ (1 + E)y]$$

This last statement is equivalent to (B) if and only if:

$$(1 - E)y = a - d \quad \text{and} \quad (1 + E)y = a + d$$

Solving the system we get:

$$y = a \quad \text{and} \quad E = d / a$$

So the statements (A) and (B) are equivalent to:

$$\text{(C)} \qquad x = (1 + \varepsilon)a \quad \text{with} \quad |\ \varepsilon\ | \leqslant d / a$$

(1.50) <u>Theorem</u> (stability of the *bisezione* procedure).

Consider the implementation in *Scilab*[1], of the *bisezione* procedure.

<u>If</u> the assignment

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[z,v,info] = bisezione(f,a,b,delta)
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terminates with info = 0 or info = 1, then:

$$|\ z - \alpha^*\ | \leqslant \text{delta}$$

where α^* is a zero of a function g 'close' to the function f in the sense that:

for every x in [a,b] it is |f(x) - g(x)| 'small'

Informally: if info = 0 or info = 1 then the procedure returns a *good approximation* of a zero of a function *close* to the one under consideration.

(Proof omitted.)

(1.51) <u>Remark</u> (conditioning of the zeros of a regular function).

Let f:[a,b] → R be regular (differentiable with continuous f') with f' $\neq$ 0 and f(a)f(b) < 0, α be the only zero of f in [a,b], g:[a,b] → R be continuous and 'close' to f, precisely such that:

for every x in [a,b] it is $|f(x) - g(x)| \leqslant d$ with a 'small' d and $d < \min\{|f(a)|,|f(b)|\}$

By the assumptions made, g has *at least one* zero in [a,b]. We want to know *how far the zero* α *of* f *can be from a zero of* g.

Let α^* be a zero of g in [a,b]. Then we have (using Lagrange's Theorem):

$$f(\alpha^*) = f(\alpha^*) - f(\alpha) = f'(t)(\alpha^* - \alpha) \quad \text{with} \quad t \text{ between } \alpha^* \text{ and } \alpha$$

Hence, denoted *m* = min{ |f'(x)|, x in [a,b] }, we have:

$$|\alpha^* - \alpha| = \frac{|f(\alpha^*)|}{|f'(t)|} \leqslant \frac{|f(\alpha^*)|}{m}$$

1 See Statement (1.08), Lecture 2.

Finally, since:

$$|f(\alpha^*)| = |f(\alpha^*) - g(\alpha^*)| \leqslant d$$

we get:

$$|\alpha^* - \alpha| \leqslant \frac{d}{m}$$

The quantity 1/m plays the role of *condition number*: the larger it is, the further the zeros of g can be from the zero of f.

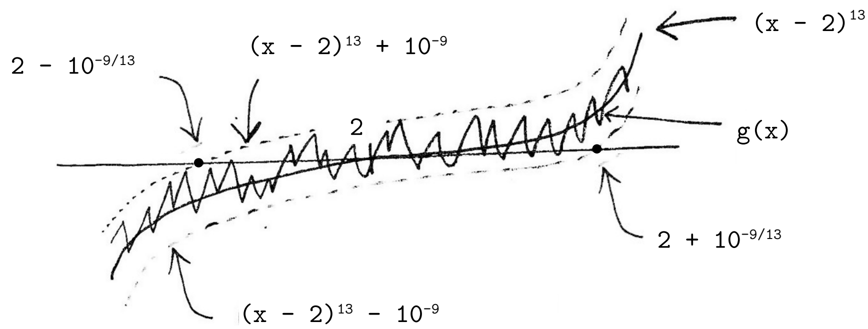
If f'(x) = 0 for some x in [a,b], in particular if $f'(\alpha) = 0$, the conditioning is certainly *bad*, as highlighted in the following Example.

(1.52) Example.

Let $f(x) = (x - 2)^{13}$. The function has only one zero, $\alpha = 2$, and is regular in the interval [1,3]. Then consider g:[1,3] → R continuous such that:

$$\text{for every x in [1,3] it is } |f(x) - g(x)| \leqslant 10^{-9}$$

The graph of such a function g is shown in the figure.



In the *worst case* the distance between the zero α of f and a zero α^* of g is $10^{-9/13} \approx 0.2$, *much larger* than the distance 10^{-9} between f and g.

(1.1) ONE-POINT METHODS

The strength of the bisection method is its generality: it can be applied to *any* function that is continuous and takes values of opposite signs at the endpoints of an interval. However, in some applications, the method requires an *excessive number of iterations* to achieve the accuracy required by the user. To overcome this drawback, let's analyze other methods for approximating the zero of a function: *one-point methods*.

(1.53) Definition (one-point method).

Let h:[a,b] → R be a continuous function. *The one-point method defined by h* is the following procedure:

z = *MetodoUnPunto*(h,a,b,γ)

<u>input</u>: a continuous h:[a,b] → R, γ in [a,b]

- x(0) = γ;
- <u>for</u> k = 1,2,3,... <u>repeat</u>
  <u>if</u> x(k-1) in [a,b] <u>then</u> x(k) = h(x(k-1)) <u>else</u> STOP

<u>output</u>: when an appropriate *stopping criterion* is met: z = x(k).

(1.54) <u>Remark</u>.

If we omit the stopping criterion and for every k we have x(k-1) in [a,b], the one-point method defines a sequence x(0), x(1), x(2),... <u>If</u> the sequence is *convergent*, its limit is a *fixed point* of h.[2]

(Proof. The sequence x(0), x(1), x(2),... is identical to the sequence h(x(0)), h(x(1)), h(x(2)),... Therefore the latter is convergent and, if α be the limit of the sequence x(k):

$$\lim_{k\to\infty} h(x(k)) = \alpha$$

Since h is a continuous function and the sequence x(k) converges to α, we have:

$$\lim_{k\to\infty} h(x(k)) = h(\lim_{k\to\infty} x(k)) = h(\alpha)$$

By the uniqueness of the limit of a convergent sequence, we deduce that α = h(α).)

(1.55) <u>Remark</u>.

Let f be the continuous function for which we are interested in approximating some zero. As stated in the previous Remark, ‘if all goes well’ the one-point method defined by h can be used to approximate *a fixed point* of h. The one-point method can be used to approximate some zero of f, if we *choose* the function h that defines it such that:

(#) {zeroes of f} = {fixed point of h}

The question arises whether *there exist* (continuous) functions h with the required property.

Consider the function h defined as follows:

$$h(x) = f(x) + x$$

If α is a zero of f, i.e. f(α) = 0, we have:

$$h(\alpha) = f(\alpha) + \alpha = \alpha \;\Rightarrow\; \alpha \text{ is a fixed point of h}$$

Conversely, if α is a fixed point of h (i.e. α = h(α)), we have:

2 The real number α is a *fixed point of* h meaning that α = h(α).

$$h(\alpha) = f(\alpha) + \alpha \quad \Rightarrow \quad f(\alpha) = 0 \quad \Rightarrow \quad \alpha \text{ is a zero of } f$$

The function h is therefore *a* function that verifies the property (#).

It is easily verified that, if g is a continuous function such that $g(x) \neq 0$ for all x, the function h defined by:

$$h(x) = g(x)f(x) + x$$

it is continuous and satisfies the property (#). So there exist *infinitely many* functions h that have as their fixed points all and only the zeros of f.

The problem now arises of choosing, among all the possible functions that have the property (#), an h such that the method defined by it generates a *convergent* sequence.