Lecture 7 (hrs. 11,12) - October 7, 2025, 16:30 - 18:30 F3

(1.44) Remark (stability, non-elementary case).

Let $f_1,f_2:R \to R$ be two elementary functions and $\varphi_1,\varphi_2:R \to M$ be the algorithms used to approximate, respectively, the values of f_1 and f_2. Let $x \in R$, $f(x) = f_2(f_1(x))$ and $\varphi(x) = \varphi_2(\varphi_1(x))$. Finally, let the algorithms φ_1 and φ_2 be *stable* on R. The question is whether the algorithm φ is stable when used to approximate f in x.

Using the stability of φ_1 and φ_2 we have: there exist real numbers $\varepsilon_1,\dots,\varepsilon_4$ such that $|\varepsilon_j| \leqslant u$, $j = 1,2,3,4$ and:

$$\varphi(x) = \varphi_2(\varphi_1(x)) = (1 + \varepsilon_4)f_2(\ (1 + \varepsilon_3)(1 + \varepsilon_1)f_1((1 + \varepsilon_2)x)\ )$$

Let $(1 + \varepsilon_3)(1 + \varepsilon_1) = 1 + t$, i.e. $t = \varepsilon_3 + \varepsilon_2 + \varepsilon_2\varepsilon_3$. It is: $|t| \leqslant 2u + u^2$ (< 1) and

$$\varphi(x) = (1 + \varepsilon_4)f_2(\ (1 + t)f_1((1 + \varepsilon_2)x)\ )$$

Let ϑ be the relative error committed when approximating $f_2(\ f_1((1 + \varepsilon_2)x)\ )$ with $f_2(\ (1 + t)f_1((1 + \varepsilon_2)x)\ )$. We rewrite:

$$f_2(\ (1 + t)f_1((1 + \varepsilon_2)x)\ ) = (1 + \vartheta)f_2(\ f_1((1 + \varepsilon_2)x)\ )$$

Hence

$$\varphi(x) = (1 + \varepsilon_4)(1 + \vartheta)f_2(\ f_1((1 + \varepsilon_2)x)\ )$$

Finally, set $(1 + \varepsilon_4)(1 + \vartheta) = 1 + \varepsilon_v$ and $\varepsilon_2 = \varepsilon_a$. We get:

$$\varphi(x) = (1 + \varepsilon_v)f((1 + \varepsilon_a)x)$$

To be able to deduce the stability of φ when used to approximate f at x, we need to investigate the magnitude of the perturbations ε_v and ε_a. Regarding ε_a we have $|\varepsilon_a| \leqslant u$, therefore ε_a is 'small'. The magnitude of ε_v, however, *depends* on that of ϑ which, in turn, *depends* on the conditioning of the calculation of f_2 in $f_1((1 + \varepsilon_2)x)$. If this latter calculation is *well conditioned* (therefore ϑ is 'small') then φ is stable when used to approximate f at x, otherwise nothing can be said about the stability of φ.

(1.45) Remark (conditioning of the calculation of regular functions).

Let $f:A \to R$ be a *regular* function (i.e. with a continuous first derivative), and $x \in A$ such that $f(x) \neq 0$. We want to study the conditioning of the calculation of f in x.

Since $f(x) \neq 0$, as stated in Remark (1.41) of Lesson 6, we must study, given a 'small' $\alpha \in$ R, the quantity:

$$\varepsilon_V = \frac{f(\,(1+\alpha)x\,) - f(x)}{f(x)}$$

By the regularity of f, using Lagrange's Theorem, we have:

there exists a real number ϑ between x and $(1 + \alpha)x$ such that

$$f(\,(1+\alpha)x\,) - f(x) = f'(\vartheta)\ \alpha\, x$$

So we rewrite:

$$\varepsilon_V = \frac{f'(\vartheta)\ \alpha\, x}{f(x)}$$

Since we have a 'small' α, we can reasonably approximate $\vartheta \approx x$ and finally rewrite:

$$\varepsilon_V \approx \frac{f'(x)}{f(x)}\ x\ \alpha$$

Introducing the *condition number* of the calculation of f in x:

$$c(x) = \left|\frac{f'(x)}{f(x)}x\right|$$

we get:

$$|\ \varepsilon_V\ | \approx c(x)\ |\ \alpha\ |$$

and the conditioning of the computation of f at x depends only on the size of the condition number c(x).

(1.46) Example.

Let f(x) = sin(x) and $x \in (0, \pi/2)$. The condition number of the computation of f at x is:

$$c(x) = \left|\frac{\cos(x)}{\operatorname{sen}(x)}x\right| = \left|\frac{x}{\tan(x)}\right| = \frac{x}{\tan(x)} < 1$$

So in this case the calculation of sin(x) is *well conditioned*. But if we consider x *close to* (but not equal to) π, taking into account that:

$$\lim_{t\to\pi} c(x) = \lim_{t\to\pi}\left|\frac{x}{\tan(x)}\right| = +\infty$$

the calculation of sin(x) is *not* well conditioned.

(1.47) <u>Remark</u> (conditioning of arithmetic operations).

Let $f(x_1,x_2) = x_1 + x_2$ and let x_1, x_2 be such that $f(x_1,x_2) \neq 0$. We want to study the conditioning of the computation of f in x_1, x_2.

Since $f(x_1, x_2) \neq 0$, as stated in Remark (1.41) of Lecture 6, we must study, given 'small' real numbers α_1 and α_2, the quantity:

$$\varepsilon_V = \frac{(1+\alpha_1)\,x_1 + (1+\alpha_2)\,x_2 - (x_1+x_2)}{x_1+x_2} = \frac{x_1}{x_1+x_2}\,\alpha_1 + \frac{x_2}{x_1+x_2}\,\alpha_2$$

Introducing the condition numbers:

$$c_1(x_1,x_2) = \left|\frac{x_1}{x_1+x_2}\right| \quad \text{and} \quad c_1(x_1,x_2) = \left|\frac{x_2}{x_1+x_2}\right|$$

we get:

<u>if</u> $x_1x_2 > 0$ (that is, the two addends have the same sign) <u>then</u>:

$$c_1(x_1,x_2) < 1 \quad \text{and} \quad c_2(x_1,x_2) < 1$$

and the conditioning of the calculation of the sum is *good*. However, if $x_1x_2 < 0$ (i.e. the two addends have opposite signs), the conditioning of the calculation can be *worse the smaller* $x_1 + x_2$ is. In fact, given $x_1 \neq 0$ and set $x_2 = y - x_1$ (i.e. $x_1 + x_2 = y$) with $y \neq 0$, we get:

$$c_1(x_1,x_2) = \left|\frac{x_1}{y}\right| \quad , \quad c_1(x_1,x_2) = \left|1-\frac{x_1}{y}\right|$$

and:

$$\lim_{y\to 0} c_1(x_1,x_2) = +\infty \quad , \quad \lim_{y\to 0} c_2(x_1,x_2) = +\infty$$

In the case of the other arithmetic operations we have:

$$\varepsilon_V = \alpha_1 + \alpha_2 + \alpha_1\,\alpha_2 \quad \text{(multiplication)}$$

$$\varepsilon_V = \frac{\alpha_1-\alpha_2}{1-\alpha_2} \quad \text{(division)}$$

and in both cases the calculation is *always well conditioned.*