Lecture 02 (hrs. 01,02) - 24 September 2025, 11:30 - 13:30 A13

(1) FUNCTION ZEROES AND COMPUTER ARITHMETIC

(1.01) The problem.

Given a continuous function f:[a,b] → R such that there exists t in R s.t. f(t) = 0, determine t. The number t is called ‘a zero of f’.

(1.02) Theorem (existence of a zero)

Let f:[a,b] → R be a continuous function s.t. f(a)f(b) < 0. Then: there exists t in (a,b) s.t. f(t) = 0.

(1.03) Remark.

The condition f(a)f(b) < 0 is equivalent to the condition:

f(a) is *not = 0* & f(b) is *not = 0* & sign f(a) *not equal to* sign f(b)

(1.04) Bisection method.

Idea: use Theorem (1.02) to obtain a sequence of intervals I(k) = [a(k),b(k)] such that:

- for every k, there exists a zero of f in I(k)
- I(k+1) is a subset of I(k)
- when k → ∞ it is measure I(k) → 0

(1.05) Description of the method.

z = *Bisection*(f,a,b)

input: f:(a,b) → R s.t. f(a)f(b) < 0

- a(0) = a; b(0) = b; I(0) = [a(0),b(0)]; x(0) = (a(0) + b(0)) / 2;
- for k = 1,2,3,… repeat:
  - if f(x(k-1)) = 0 then STOP; else
    - if f(x(k-1))f(b(k-1)) < 0 then a(k) = x(k-1); b(k) = b(k-1);
      else a(k) = a(k-1); b(k) = x(k-1);
    - I(k) = [a(k),b(k)]; x(k) = (a(k) + b(k)) / 2;

output: when a suitable *halt condition* is met: z = x(k), midpoint of the last interval.

(1.06) Remark.

(A) measure I(k) = b(k) - a(k) = measure I(k-1) / 2^1 = measure I(k-2) / 2^2 = ... = measure I(0) / 2^k and then:

when k → ∞ it is measure I(k) → 0

(B) if f is a continuous function then: for every k, I(k) contains a zero of f and

$$\text{when } k \to \infty \text{ it is } x(k) \to t, \text{ where } f(t) = 0$$

(Proof ...)

(1.07) Halt condition.

The bisection method is an *iterative method*, that is, a method that approximates the desired object by constructing a *sequence*. Since it is physically impossible to construct *all* the elements of the sequence, it is *necessary* to introduce a halt condition, that is, a condition that, when satisfied, stops the construction of the sequence.

An example of a halt condition is: given a *positive* real number Δ ...

$$\underline{\text{if}} \text{ measure } I(k) < \Delta \ \underline{\text{then}} \text{ STOP}$$

Properties of the halt condition:

(1) the *condition* measure I(k) < Δ 'is *computable*'
(2) the condition is *certainly verified* after a *finite* number of iterations (see Remark (B) in (1.06)): the criterion 'is *effective*'
(3) if f is a continuous function and k is such that measure I(k) < Δ then:

- there exists t in I(k) s.t. t is a zero di f
- |x(k) - t| < measure I(k) / 2 < Δ/2 < Δ

that is, the procedure returns a value x(k) that is an approximation of a zero of f with *absolute error* |x(k) - t| less than Δ: 'the procedure returns an approximation *as accurate as required by the user*'.