* METODI DIRETTI IN $F(\beta,m)$ *

operando in $\mathbb{R}$	operando in $F(\beta,m)$
① $(S,D,P) = EGP(A)$	① $(\hat{S}, \hat{D}, \hat{P}) = \widehat{EGP}(\hat{A})$
② $c = SA(S, Pb)$	② $\hat{c} = \widehat{SA}(\hat{S}, \hat{P}\hat{b})$
③ $x_* = SI(D,c)$	③ $\hat{x} = \widehat{SI}(\hat{D}, \hat{c})$

Oss: le procedure $\widehat{EGP}$ e $\widehat{SA}$ operano con dati $\hat{A}$, $\hat{b}$ potenzialmente diversi dai corrisp A, b. Infatti qualche elem di A e b potrebbe non essere in $F(\beta,m)$.

Es 1: Supponiamo: $\hat{A} = rd(A)$, $\hat{b} = rd(b)$.

- $\hat{a}_{ij} = rd(a_{ij}) = a_{ij}(1+\varepsilon_{ij})$ con $|\varepsilon_{ij}| \leq u$

  $\Rightarrow \hat{A} = A + E$, $e_{ij} = a_{ij}\varepsilon_{ij}$ $\Rightarrow \boxed{\|E\|_1 \leq u\|A\|_1}$

- $\hat{b}_j = rd(b_j) = b_j(1+\varepsilon_j)$ con $|\varepsilon_j| \leq u$

  $\Rightarrow \hat{b} = b + f$, $f_j = b_j\varepsilon_j$ $\Rightarrow \boxed{\|f\|_1 \leq u\|b\|_1}$

- Consideriamo il sist $\hat{A}x = \hat{b}$, ovvero

$$(A+E)x = b+f$$

$$\varepsilon_b = \frac{\|f\|_1}{\|b\|_1} \leq u \quad , \quad \varepsilon_A = \frac{\|E\|_1}{\|A\|_1} \leq u \quad \Rightarrow \quad \alpha = u$$

<u>SE</u> $u \, c_1(A) < \frac{1}{10}$ <u>allora</u>

- $\hat{A} = A + E$ invertibile
- $$\boxed{\varepsilon_x = \frac{\|\hat{x} - x_*\|_1}{\|x_*\|_1} \leq 2 u \, c_1(A)}$$

Operando in $F(\beta, m)$ NON È RAGIONEVOLE aspettarsi ris migliori di

$$\varepsilon_x \leq 2 u \, c_1(A)$$

<u>Es 2</u>: Operiamo in $F(\beta, m)$ supponendo che:

(1) $\hat{A} = A$, $\hat{b} = b$ (i dati sono in $F(\beta, m)$)

(2) $\widehat{EGP}(A) = (S, D, P)$ (la fatt LR è esatta)

(3) $\widehat{SA}(S, Pb) = \hat{c} = rd(c)$

(4) $\widehat{SI}(D, \hat{c}) = SI(D, \hat{c})$ (la procedura $\widehat{SI}$ opera "senza errori")

- $\hat{x} = SI(D, \hat{c})$, $x^* = SI(D, c)$

- $\varepsilon_c = \dfrac{\|\delta c\|_1}{\|c\|_1} \leqslant u \equiv \alpha$

$\underline{\text{SE}}$ $\quad u\, c_1(D) < \dfrac{1}{10}$ $\quad \underline{\text{allora}}$ $\quad \dfrac{\|\hat{x} - x^*\|_1}{\|x^*\|_1} \leqslant 2u\, c_1(D)$

- $c_1(D) = c_1(A) \boxed{\dfrac{c_1(D)}{c_1(A)}} \leftarrow$ "fattore di amplificazione" del n di condiz

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<u>Es</u>: $\quad \gamma \in (0,1)$ e $A = \begin{bmatrix} \gamma & 1 \\ 1 & 0 \end{bmatrix}$

- $\|A\|_1 = 1+\gamma$

- $A^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -\gamma \end{bmatrix} \Rightarrow \|A^{-1}\|_1 = 1+\gamma \Rightarrow c_1(A) = (1+\gamma)^2 < 4$

- $\text{EGP}(A) = \Bigg(\underset{S}{\begin{bmatrix} 1 & 0 \\ 1/\gamma & 1 \end{bmatrix}}, \underset{D}{\begin{bmatrix} \gamma & 1 \\ 0 & -1/\gamma \end{bmatrix}}, \underset{P}{I} \Bigg)$

- $D^{-1} = \begin{bmatrix} 1/\gamma & 1 \\ 0 & -\gamma \end{bmatrix} \Rightarrow c_1(D) = \left(1 + \frac{1}{\gamma}\right) \max\left\{1+\gamma, \frac{1}{\gamma}\right\}$

e $\quad \displaystyle\lim_{\gamma \to 0} \frac{c_1(D)}{c_1(A)} = +\infty$

(scegliendo γ opportunam PICCOLO si ottiene $\dfrac{c_1(D)}{c_1(A)}$ arbitrariam GRANDE)

Il procedim riduce la soluz di $Ax = b$ a quello di $Dx = c$, ma i due sistemi, benché EQUIVALENTI (hanno la stessa soluz) NON hanno le stesse propr di condizionam:

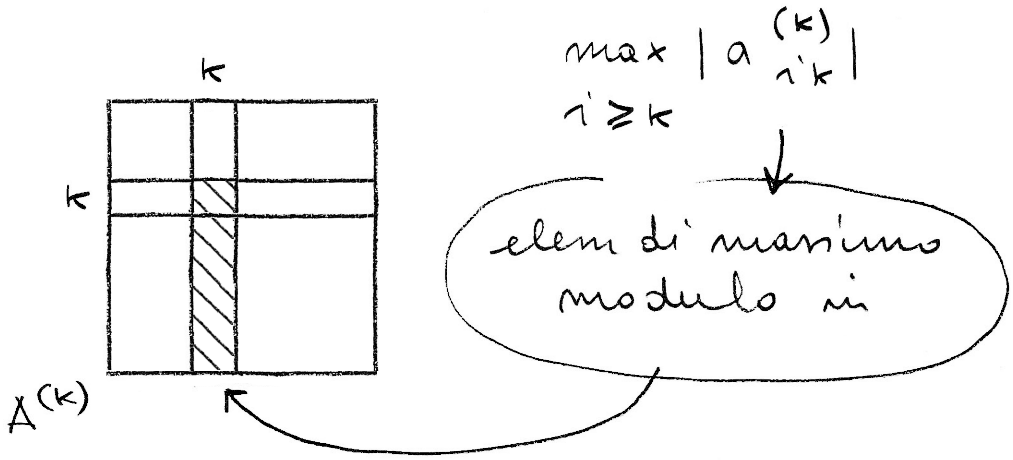
$$\frac{c(D)}{c(A)} \gg 1 \;!$$

Il procedim, SODDISFACENTE op in $\mathbb{R}$, è NON SODDISFACENTE op in $F(\beta, m)$!

Rimedio: EGPP

(Elim di Gauss con Pivoting Parziale)

al passo k si utilizza come pivot l'elemento



Es:

$$\underset{A}{\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}} \xrightarrow{P_{12}} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow{H_1} \begin{bmatrix} 2 & 1 & -1 \\ 0 & -1/2 & 3/2 \\ 0 & 3/2 & 3/2 \end{bmatrix} - \cdots$$

$$\cdots \xrightarrow{P_{32}} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3/2 & 3/2 \\ 0 & -1/2 & 3/2 \end{bmatrix} \xrightarrow{H_2} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3/2 & 3/2 \\ 0 & 0 & 2 \end{bmatrix} = D$$

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & -1/3 & 1 \end{bmatrix}, \quad P = P_{32} P_{12} \; ; \qquad \boxed{EGPP(A) = (S, D, P)}$$

<u>Oss</u>: utilizz EGPP:

$\forall A \in \mathbb{R}^{n \times n}$ invertibile:

$$\boxed{\frac{c(D)}{c(A)} \leq F(n)}$$

dip <u>solo</u> dalla dim della matrice

in particolare NON si può avere crescita illimitata del n. di cond nel passaggio da A a D.

<u>dim</u>:

$$\frac{c_1(D)}{c_1(A)} = \frac{\|D\|_1 \cdot \|D^{-1}\|_1}{\|A\|_1 \cdot \|A^{-1}\|_1} = \frac{\|S\|_1 \cdot \|D\|_1}{\|A\|_1} \cdot \frac{\|D^{-1}\|_1}{\|A^{-1}\|_1 \|S\|_1}$$

Poichè:

(i) $PA = SD \Rightarrow D^{-1} = A^{-1} P^T S$ e

(ii) $\forall M, N \in \mathbb{R}^{n \times n}: \quad \|MN\|_1 \leq \|M\|_1 \cdot \|N\|_1$

allora:

perche' P^T di perm!

$$\frac{\| D^{-1}\|_1}{\|A^{-1}\|_1 \cdot \|S\|_1} \leqslant \frac{\|A^{-1}\|_1 \cdot \|P^T\|_1 \cdot \|S\|_1}{\|A^{-1}\|_1 \, \|S\|_1} = \|P^T\|_1 = 1$$

ovvero:
$$\boxed{\frac{c_1(D)}{c_1(A)} \leqslant \frac{\|S\|_1 \cdot \|D\|_1}{\|A\|_1}}$$

- con EGPP: $\dfrac{\|S\|_\infty \, \|D\|_\infty}{\|A\|_\infty} \leqslant n\, 2^{n-1}$

$$\Rightarrow \frac{\|S\|_1 \, \|D\|_1}{\|A\|_1} \leqslant n^2\, 2^{n-1}$$

[<u>dim</u>: (i) $|s_{ij}| \leqslant 1 \Rightarrow \|S\|_\infty \leqslant n$ e $\|S\|_1 \leqslant n$;

(ii) $D = H_{n-1} P_{n-1} \cdots H_1 P_1 A$

$$\Rightarrow \|D\|_\infty \leqslant \|H_{n-1}\|_\infty \cdots \|H_1\|_\infty \, \|A\|_\infty$$
$$\leqslant 2^{n-1} \|A\|_\infty \quad [\forall k, \ \|H_k\|_\infty \leqslant 2]$$

(iii) $\|D\|_1 \leqslant \|H_{n-1} P_{n-1} \cdots H_1 P_1\|_1 \, \|A\|_1$

e $\|H_{n-1} P_{n-1} \cdots H_1 P_1\|_1 \leqslant n \|H_{n-1} P_{n-1} \cdots H_1 P_1\|_\infty$

$$\leqslant n \, \|H_{n-1}\|_\infty \cdots \|H_1\|_\infty \leqslant n\, 2^{n-1}$$

$$\Rightarrow \|D\|_1 \leqslant n\, 2^{n-1} \|A\|_1$$]

- uso della fatt QR in $F(\beta,m)$:

operando in $\mathbb{R}$	operando in $F(\beta,m)$
① $(U,T) = qr(A)$	① $(\hat{U},\hat{T}) = \widehat{qr}(\hat{A})$
② $c = U^T b$	② $\hat{c} = \hat{U}^T \otimes \hat{b}$
③ $x_* = SI(T,c)$	③ $\hat{x} = \widehat{SI}(\hat{T},\hat{c})$

Es. 1: identico al caso con fatt LR.

Es. 2: $\hat{A}=A$, $\hat{b}=b$, $\widehat{qr}(A) = qr(A) = (U,T)$,
$U^T \otimes b = \hat{c} = rd(c)$, $\widehat{SI}(T,\hat{c}) = SI(T,\hat{c})$

ecc.

(1) $A = UT \Rightarrow \|A\|_2 = \|UT\|_2 = \max_{\|v\|_2=1} \|U(Tv)\|_2$

$$= \max_{\|v\|_2=1} \|Tv\|_2$$

$$= \|T\|_2$$

ortogonale $\Rightarrow \forall w$, $\|Uw\|_2 = \|w\|_2$

(2) $T^{-1} = A^{-1}U \Rightarrow \|T^{-1}\|_2 = \|A^{-1}U\|_2 = \max_{\|v\|_2=1} \|A^{-1}(Uv)\|_2$

$$= \max_{\|Uv\|_2=1} \|A^{-1}(Uv)\|_2 = \|A^{-1}\|_2$$

Q.d.: $\boxed{\dfrac{c_2(T)}{c_2(A)} = 1}$

$\Rightarrow c_1(T) = \|T\|_1 \|T^{-1}\|_1 \leq n \|T\|_2 \|T^{-1}\|_2 = n \|A\|_2 \|A^{-1}\|_2$

$$\leq n^2 \|A\|_1 \|A^{-1}\|_1 = n^2 c_1(A) \Rightarrow \boxed{\frac{c_1(T)}{c_1(A)} \leq n^2}$$