$x^* \mid Ax^* = b$ soluz del sist non perturbato

$\hat{x} \mid (A+\delta A)\hat{x} = b + \delta b$ soluz del sist perturbato

PERTURBAZ dei dati

$\delta x = \hat{x} - x^* = F(A, b; \delta A, \delta b)$

- CASO 1: $\delta A = 0$, $b \neq 0$

$\hat{x} = A^{-1}(b+\delta b)$; $\delta x = A^{-1}\delta b$

$\|\delta x\| = \|A^{-1}\delta b\|$; $\boxed{\varepsilon_d = \dfrac{\|\delta x\|}{\|x^*\|}}$ ERRORE RELATIVO

$$\varepsilon_d = \frac{\|A^{-1}\delta b\|}{\|A^{-1}b\|} \leq \|A^{-1}\| \frac{\|\delta b\|}{\|b\|} \frac{\|b\|}{\|A^{-1}b\|} \leq \|A^{-1}\| \|A\| \frac{\|\delta b\|}{\|b\|}$$

$\exists\, \delta b$ t.c. $=$ (dalla def di $\|A^{-1}\|$)

$\boxed{\varepsilon_b = \dfrac{\|\delta b\|}{\|b\|}}$

$\exists\, b$ t.c. $=$

Oss: $\inf\left\{ \dfrac{\|A^{-1}x\|}{\|x\|}, x \neq 0 \right\} \leq \dfrac{\|A^{-1}b\|}{\|b\|}$

$\| \quad \min\{\|A^{-1}x\|, \|x\|=1\}$

perciò: $\dfrac{\|b\|}{\|A^{-1}b\|} \leq \left(\min\{\|A^{-1}x\|, \|x\|=1\}\right)^{-1}$

$\| \quad \|A\|$

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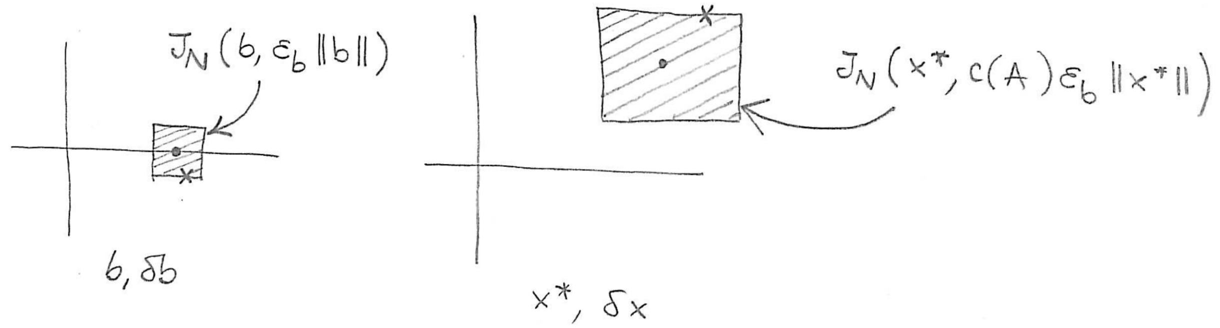
def (numero di condiz): $A \in \mathbb{R}^{n\times n}$, invertibile;

$c(A) = \|A\| \|A^{-1}\|$ NUMERO di CONDIZIONAM di A

$\boxed{\text{TEO (condiz, I): } A \in \mathbb{R}^{n\times n}, \text{ invert;} \quad \bullet\ \forall b \in \mathbb{R}^n, b \neq 0,\ \forall \delta b \in \mathbb{R}^n:\ \varepsilon_d \leq c(A)\,\varepsilon_b \quad \bullet\ \exists b \in \mathbb{R}^n, b \neq 0,\ \exists \delta b \in \mathbb{R}^n:\ \varepsilon_d = c(A)\,\varepsilon_b}$

$c(A)$

Oss:



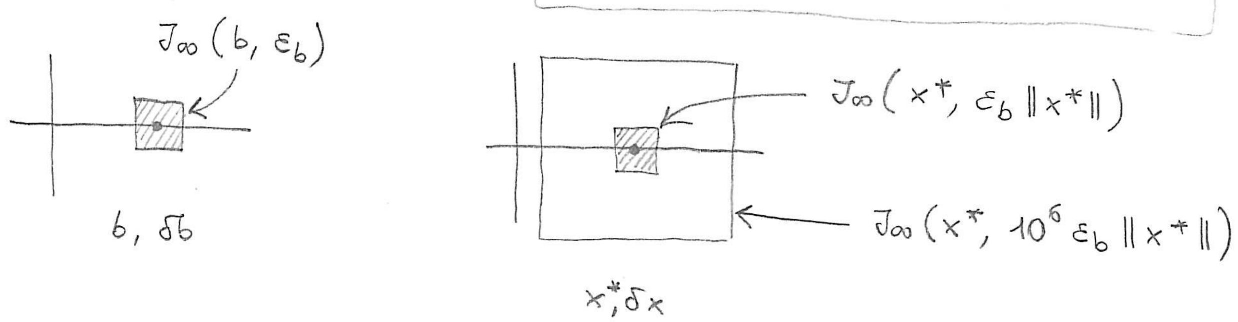
Es: $A = \begin{bmatrix} 10^{-3} & 0 \\ 0 & 10^3 \end{bmatrix}$ • $\det A = 1$, $c_\infty(A) = 10^6$

• $x^* = \begin{bmatrix} 10^3 b_1 \\ 10^{-3} b_2 \end{bmatrix}$, $\hat{x} = x^* + \delta x = \begin{bmatrix} 10^3(b_1+\delta b_1) \\ 10^{-3}(b_2+\delta b_2) \end{bmatrix}$, $\delta x = \begin{bmatrix} 10^3 \delta b_1 \\ 10^{-3}\delta b_2 \end{bmatrix}$

(1) $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow x^* = \begin{pmatrix} 10^3 \\ 0 \end{pmatrix}$, $\|x^*\|_\infty = 10^3$

• $\varepsilon_b = \dfrac{\|\delta b\|_\infty}{\|b\|_\infty} = \|\delta b\|_\infty$

• $\|\delta x\|_\infty \leq 10^3 \|\delta b\|_\infty \Rightarrow \boxed{\varepsilon_d \leq \dfrac{10^3 \|\delta b\|_\infty}{10^3} = \|\delta b\|_\infty = \varepsilon_b}$

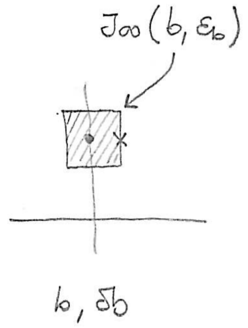


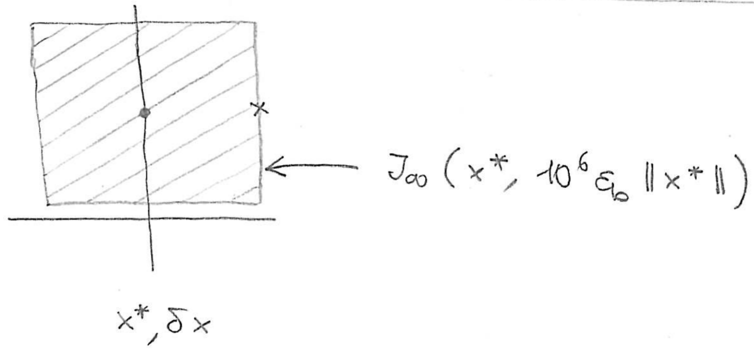
(2) $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\delta b = \begin{pmatrix} \delta b_1 \\ 0 \end{pmatrix} \Rightarrow x^* = \begin{pmatrix} 0 \\ 10^{-3} \end{pmatrix}$, $\|x^*\|_\infty = 10^{-3}$,

$\delta x = \begin{pmatrix} 10^3 \delta b_1 \\ 0 \end{pmatrix}$

- $\varepsilon_b = \|\delta b\|_\infty$

- $\|\delta x\|_\infty = 10^3 \|\delta b\|_\infty \Rightarrow \boxed{\varepsilon_d = \dfrac{10^3 \|\delta b\|_\infty}{10^{-3}} = 10^6 \|\delta b\|_\infty = 10^6 \varepsilon_b}$





- <u>CASO 2</u>: $\delta b = 0$, δA t.c. $A + \delta A$ invert

$\boxed{\varepsilon_A = \dfrac{\|\delta A\|}{\|A\|}}$ $\boxed{\hat{\varepsilon}_d = \dfrac{\|\delta x\|}{\|\hat{x}\|}}$

<u>TEO</u> (condiz, II): $A \in \mathbb{R}^{n \times n}$, invert;

- $\forall b \begin{cases} \in \mathbb{R}^n \\ \neq 0 \end{cases}$, $\forall \delta A \begin{cases} \in \mathbb{R}^{n \times n} \\ \text{t.c. } A + \delta A \text{ invert} \end{cases}$: $\hat{\varepsilon}_d \leq c(A)\, \varepsilon_A$

- $\exists b \begin{cases} \in \mathbb{R}^n \\ \neq 0 \end{cases}$, $\exists \delta A \begin{cases} \in \mathbb{R}^{n \times n} \\ \text{t.c. } A + \delta A \text{ invert} \end{cases}$: $\hat{\varepsilon}_d = c(A)\, \varepsilon_A$

<u>Oss</u>: $(\mathbb{R}^n, N)$, $A \in \mathbb{R}^{n \times n}$ invert : $c_N(A) \geq 1$

(<u>dim</u>: $I = A^{-1}A \Rightarrow 1 = \|I\|_N \leq \|A^{-1}\|_N \|A\|_N$)

<u>Es</u>: $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$; $b, \delta A$ t.c. $\hat{x} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$, $\varepsilon_A \leq \dfrac{1}{100}$

$\downarrow$

$\dfrac{\|\delta A\|_2}{\|A\|_2}$

- $\|A\|_2 = \sqrt{\max\{1, 9\}} = 3$

$$\boxed{\begin{aligned} p_A(x) &= (2-x)^2 - 1 = (2-x-1)(2-x+1) \\ &= (1-x)(3-x), \quad \lambda_1 = 1, \quad \lambda_2 = 3 \\ A^TA &= A^2 \quad (\text{perchè } A \text{ è simm!}) \\ \Rightarrow &\text{ autovalori di } A^TA = \lambda_1^2, \lambda_2^2 \end{aligned}}$$

- $\|A^{-1}\|_2 = \sqrt{\max\{1, 1/9\}} = 1$

$\boxed{\underline{\text{Oss}}: A \text{ simm} \Rightarrow c_2(A) = \dfrac{\max|\lambda_k|}{\min|\lambda_k|}}$

- $c_2(A) = 3 \Rightarrow \hat{\varepsilon}_d \leq c_2(A)\, \varepsilon_A \leq \dfrac{3}{100}$

$\Rightarrow \|\delta x\|_2 = \hat{\varepsilon}_d \|\hat{x}\|_2 \leq \dfrac{3}{100} \cdot 10 = \dfrac{3}{10}$