CN # 40 / 6 maggio 2014 / C43

- Velocità di convergenza

$Ax = b$, A invert, $\exists!\, x^*$ tc $Ax^* = b$

H, c t.c. $(I-H)x = c \sim Ax = b$, ovvero

$(I-H)x^* = c$; se $x_k \to x^*$, $x_k - x^* \longrightarrow 0$

e viceversa. Ma

$$x_{k+1} = Hx_k + c \iff x_{k+1} = Hx_k + x^* - Hx^*$$

$$\iff x_{k+1} - x^* = H(x_k - x^*)$$

ovvero, posto $e_k = x_k - x^*$: $\boxed{e_{k+1} = H e_k}$

iteraz per l'errore e_k

__Es__: $H = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, con $1 > |\lambda_1| > |\lambda_2|$;

$$e_k = H^k e_0 = \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix} e_0 \quad \text{da cui}$$

$$\| e_k \| = \sqrt{(\lambda_1^k e_{01})^2 + (\lambda_2^k e_{02})^2} = \quad \boxed{\text{ip: } e_{01} \neq 0}$$

$$= |\lambda_1|^k |e_{01}| \sqrt{1 + \left(\frac{\lambda_2}{\lambda_1}\right)^{2k} \left(\frac{e_{02}}{e_{01}}\right)^2}$$

dunque: $$\lim_{k \to \infty} \frac{\| e_k \|}{(\rho(H))^k} = |e_{01}| \neq 0 \qquad \boxed{\rho(H) = |\lambda_1|}$$

ovvero: $$\boxed{\| e_k \| \approx |e_{01}| (\rho(H))^k}$$

__Oss__: Se fosse $e_{01} = 0$ (e $e_{02} \neq 0$) la succes. tenderebbe a 0 PIÚ RAPIDAMENTE, ma salvo clamorosi colpi di fortuna...

__Es__: H diagonalizz.: $H = V \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} V^{-1}$

con $1 > |\lambda_1| > |\lambda_2|$

Posto $e_k' = V^{-1} e_k$ si ha:

$$e_k = H^k e_0 = \underbrace{V \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix} \overbrace{V^{-1} e_0}^{e'_0}}$$

$$\begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix} e'_0 = e'_k$$

Dall' Es precedente, SE $e'_{01} \neq 0$:

$$\| e'_k \| \approx |e'_{01}| (\rho(H))^k$$

Inoltre: $\| e_k \| = \| V e'_k \| \leq \max \| V \, \mathrm{vers}(e'_k) \| \cdot \| e'_k \|$

$$= \| V \| \, \| e'_k \|$$

e $\| e_k \| = \| V e'_k \| \geq \min \| V \, \mathrm{vers}(e'_k) \| \cdot \| e'_k \|$

$$= \frac{1}{\| V^{-1} \|} \| e'_k \|$$

Q.di: $$\frac{1}{\| V^{-1} \|} \| e'_k \| \leq \| e_k \| \leq \| V \| \, \| e'_k \|$$

$$\Rightarrow \frac{|e'_{01}|}{\| V^{-1} \|} (\rho(H))^k \lesssim \| e_k \| \lesssim \| V \| \, |e'_{01}| \, (\rho(H))^k$$

$$\boxed{\| e_k \| \approx C (\rho(H))^k}$$

- La rapidità di convergenza della successione è determinata dal <u>raggio spettrale di H</u>.

<u>Metodo di JACOBI</u>

- $A \in \mathbb{R}^{n\times n}$ $\left\langle \begin{array}{l} \text{invertibile} \\ \boxed{\text{con } a_{kk} \neq 0} \end{array}\right.$, $b \in \mathbb{R}^n$

- $A = \operatorname{diag}(a_{11}, \dots, a_{nn}) + M$

$$Ax = b \sim \operatorname{diag}(a_{11}, \dots, a_{nn})\, x = -Mx + b$$

$$\sim x = \underbrace{-\operatorname{diag}(a_{11}, \dots, a_{nn})^{-1} M}_{H_J} x + \underbrace{\operatorname{diag}(a_{11}, \dots, a_{nn})^{-1} b}_{c_J}$$

è il m. it def da H_J e c_J.

Es. $A = \boxed{\begin{matrix} 3 & -1 & 0 \\ 0 & 1 & 4 \\ 1 & 6 & 7 \end{matrix}}$, $b = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

<u>Oss</u>: $a_{kk} \neq 0$ ... $\boxed{\begin{matrix} 3 & & \\ & 1 & \\ & & 7 \end{matrix}} + \underbrace{\boxed{\begin{matrix} 0 & -1 & 0 \\ 0 & 0 & 4 \\ 1 & 6 & 0 \end{matrix}}}_{M}$

$$\Rightarrow H_J = -\boxed{\begin{matrix} 1/3 & & \\ & 1 & \\ & & 1/7 \end{matrix}} \boxed{\begin{matrix} 0 & -1 & 0 \\ 0 & 0 & 4 \\ 1 & 6 & 0 \end{matrix}} = \boxed{\begin{matrix} 0 & 1/3 & 0 \\ 0 & 0 & -4 \\ -1/7 & -6/7 & 0 \end{matrix}} , \quad c_J = \begin{bmatrix} 1/3 \\ 0 \\ 3/7 \end{bmatrix}$$