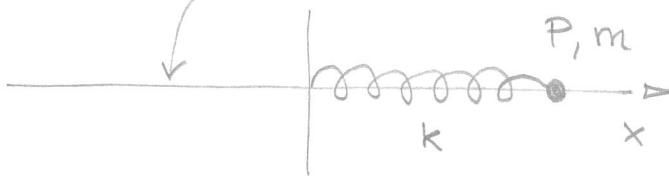


CN #49, 50 / 20 maggio 2013 / C43

qui da or'22 l'iscia

Es: Oscill armonico.



Eq. moto: $m\ddot{x} = -kx$

$$\rightarrow \ddot{x} = -\frac{k}{m}x, \quad \frac{k}{m} = 1 \text{ s}^{-2}$$

Forma normale:

$$x_1 = x$$

$$x_2 = \dot{x}$$

$$\left\{ \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 \end{array} \right.$$

Moto t.c.

$$x(0) = 1 \text{ m}$$

$$\dot{x}(0) = 0 \text{ m/s}$$

Pb. Cauchy: $X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

$$\dot{X}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} X(t), \quad X(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad t \in [0, 2\pi]$$

• $F(t, X) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X$; $F: \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$,

$$\|F(t, x_1) - F(t, x_2)\| = \left\| \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} (x_1 - x_2) \right\|$$

$$\leq \left\| \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\| \|x_1 - x_2\| = \|x_1 - x_2\|$$

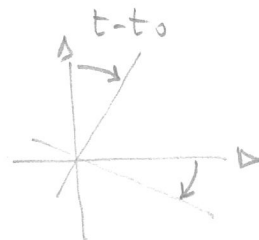
\dot{x} lip. con $L=1$ \Rightarrow $\forall t_0, x(t_0) \exists!$ soluz

- Soluzione per $X(t_0) = \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix}$:

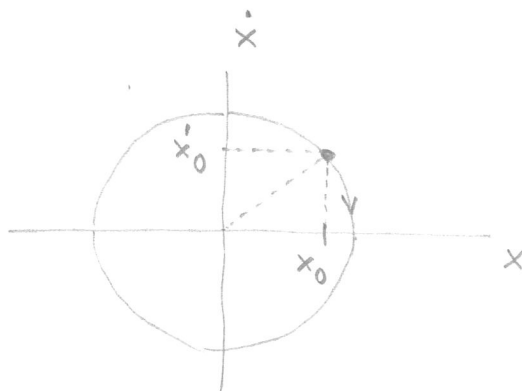
$$X(t; \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix}, t_0) = \begin{bmatrix} x_0 \cos(t-t_0) + \dot{x}_0 \sin(t-t_0) \\ -x_0 \sin(t-t_0) + \dot{x}_0 \cos(t-t_0) \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \cos(t-t_0) & \sin(t-t_0) \\ -\sin(t-t_0) & \cos(t-t_0) \end{bmatrix}}_{\text{rotazione ORARIA}} \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix}$$

rotazione ORARIA
di $t-t_0$ rad :



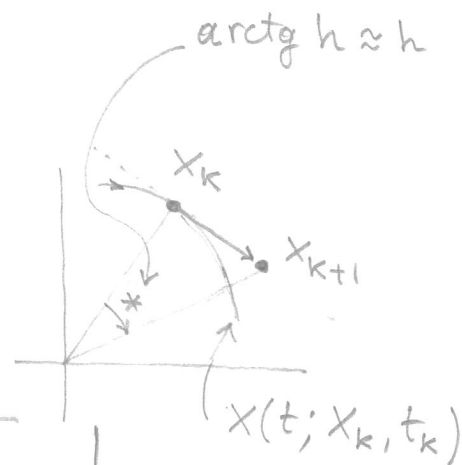
- Piano delle fasi :



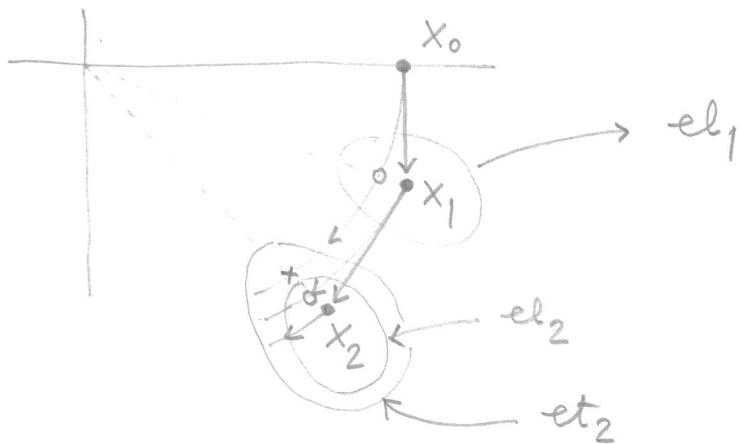
La traiett. del moto nel piano delle fasi
è una circonferenza di raggio $\sqrt{x_0^2 + \dot{x}_0^2}$
percorsa in senso orario con v. angolare
costante 1 rad/s.

- Metodo di Eulero :

$$\begin{aligned} X_{k+1} &= X_k + h F(t_k, X_k) = \\ &= X_k + h \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} X_k \end{aligned}$$



il vett $X_{k+1} - X_k$ è tg alla traiett. ...



Propagazione dell'errore locale:

$$\begin{aligned} & \| X(t_{k+1}; A, t_k) - X(t_{k+1}; B, t_k) \| \\ &= \| [\text{rot} \dots] (A - B) \| = \| A - B \| \end{aligned}$$

$$\begin{aligned} \Rightarrow & \boxed{ET_k \leq EL_k + ET_{k-1} \leq EL_k + EL_{k-1} + ET_{k-2}} \\ & \dots \leq EL_k + \dots + EL_1 \leq \boxed{kE} \end{aligned}$$

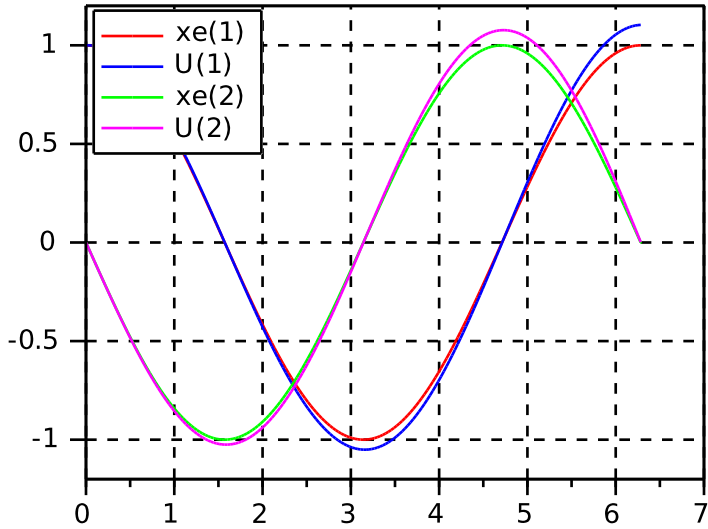
• $\| X_0 \| = \sqrt{x_0^2 + \dot{x}_0^2} = \frac{2}{m}$ (energia nel moto per x_0)

$\| X_1 \| > \| X_0 \|$: ad ogni iterazione "aumenta l'en".

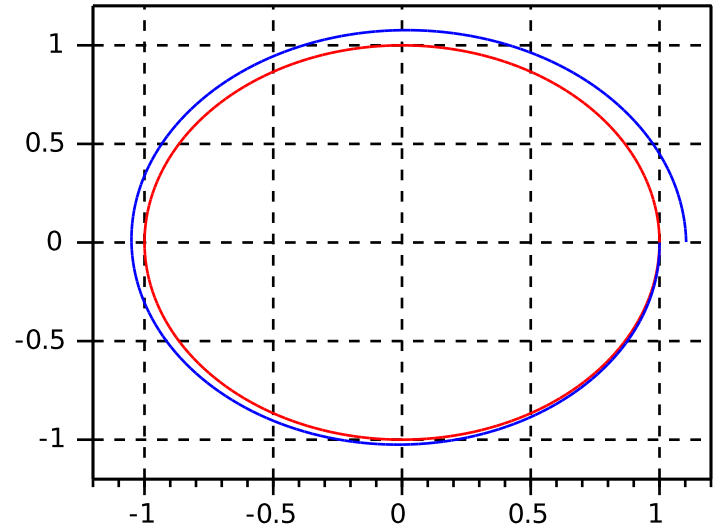
$\frac{2}{m}$ (energia nel moto per x_1)

l'attrattore sta su una traiettoria corrispondente ad un moto con energia SUPERIORE a quella del moto precedente...

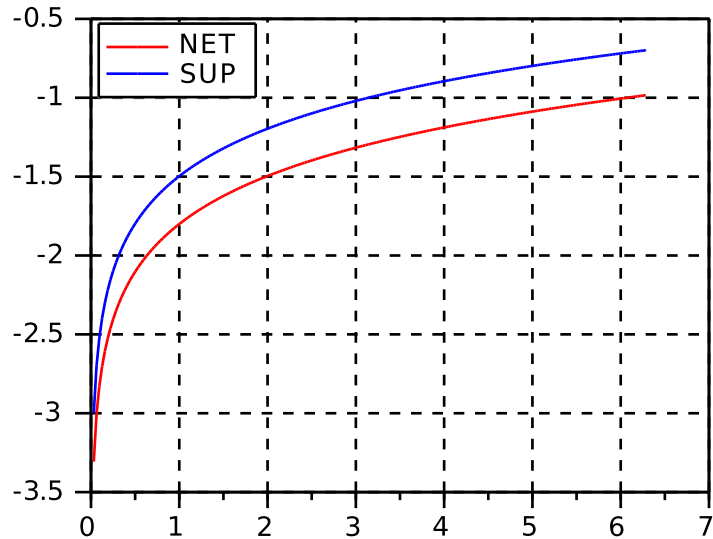
soluzione



piano delle fasi: traiettoria



log10 errore totale



Problema: oscillatore armonico ($L = 1.000D+00$)

Procedura: LMV_eulero_pv

$SUP(k) = EL_MAX + SUP(k-1)$

$EL_MAX = 1.000D-03$

Errore totale massimo = $1.037D-01$

Numero passi = $2.010D+02$

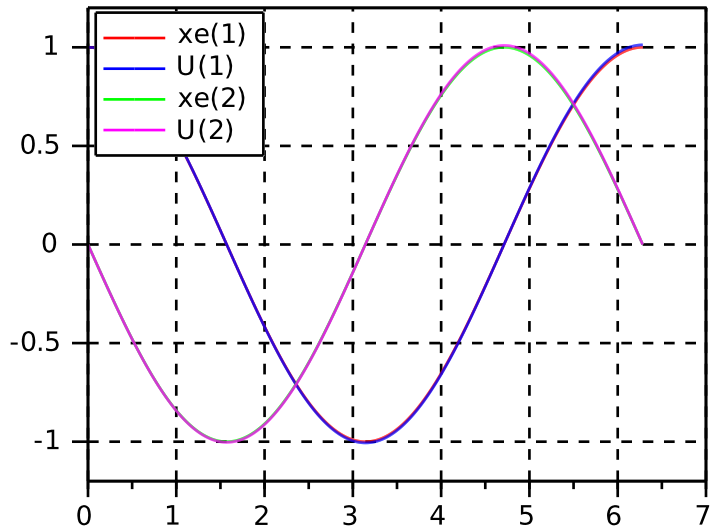
Passo:

minimo = $3.142D-02$

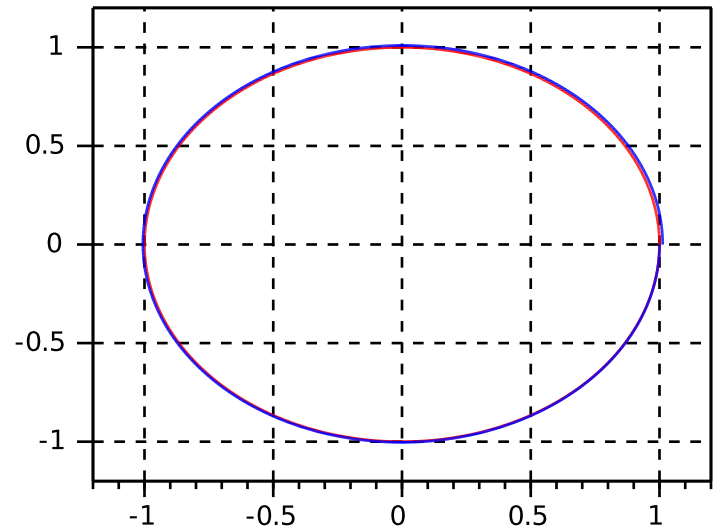
medio = $3.142D-02$

massimo = $3.142D-02$

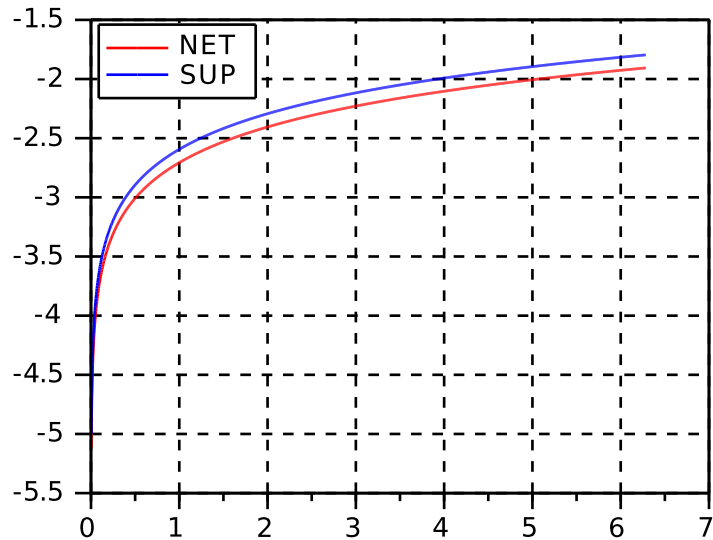
soluzione



piano delle fasi: traiettoria



log10 errore totale



Problema: oscillatore armonico ($L = 1.000D+00$)

Procedura: LMV_eulero_pv

$SUP(k) = EL_MAX + SUP(k-1)$

$EL_MAX = 1.000D-05$

Errore totale massimo = $1.241D-02$

Numero passi = $1.600D+03$

Passo:

minimo = $3.927D-03$

medio = $3.927D-03$

massimo = $3.927D-03$

Es (eq. nu STIFF)

$$\text{Pb: } \begin{cases} \dot{x} = -100x + 10 \\ x(0) = 1 \end{cases}, \quad t \in [0, 2]$$

• $F(t, x) = -100x + 10 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$

$$\begin{aligned} |F(t, x_1) - F(t, x_2)| &= |-100(x_1 - x_2)| \\ &= 100|x_1 - x_2| \end{aligned}$$

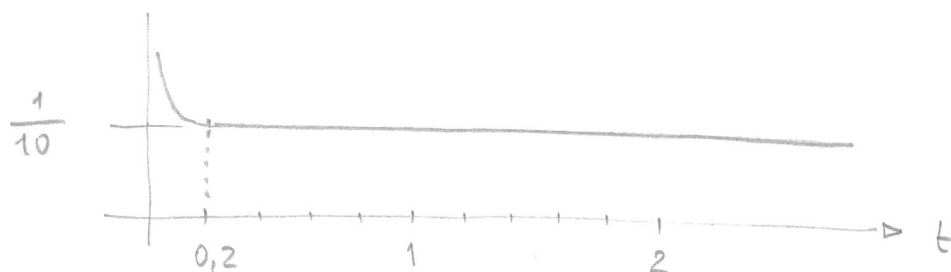
t lip con $L = 100 \Rightarrow \forall t_0, x_0 \exists!$ soluz

• Soluz per $x(t_0) = x_0$:

$$x(t) = \frac{1}{10} + \left(x_0 - \frac{1}{10}\right) e^{-100(t-t_0)}$$

$$\left[\begin{array}{l} \dot{x}(t) = -100 \left(x(t) - \frac{1}{10}\right) \\ \text{conf. eq} \\ \frac{1}{10} \end{array} \right]$$

$$\begin{aligned} t - t_0 &= 0,2 \\ \Rightarrow e^{-100(t-t_0)} &\approx 2 \cdot 10^{-9} \end{aligned}$$



- Propagez dell'err locale :

$$\left| \frac{1}{10} + \left(A - \frac{1}{10} \right) e^{-100(t-t_0)} - \right. \\ \left. - \frac{1}{10} - \left(B - \frac{1}{10} \right) e^{-100(t-t_0)} \right| =$$

$$= \left| \left(A - \frac{1}{10} - B + \frac{1}{10} \right) e^{-100(t-t_0)} \right|$$

$$= |A - B| e^{-100(t-t_0)} < |A - B|$$

$$\Rightarrow \boxed{ET_k} < EL_k + ET_{k-1} < \dots < \boxed{kE}$$