CN #46,47 / 16 maggio 2013 / A11

* Metodi numerici AD UN PASSO : aspetti QUANTITATIVI *

Pb. di Cauchy :

$$\begin{cases} \dot{x} = F(t,x) \\ x(t_0) = x_0 \end{cases} , \quad t \in [t_0, t_f]$$

• **TEOREMA** ($\exists !$ soluzione)

Sia $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ continua e

"lipschitziana"

$$\boxed{\begin{array}{c} \exists L > 0 \quad \text{t.c.} \quad \forall \ (t, x_1), (t, x_2) \in [t_0, t_f] \times \mathbb{R}^n \\ \| F(t, x_1) - F(t, x_2) \| \le L \| x_1 - x_2 \| \end{array}}$$

allora $\exists ! \ x : [t_0, t_f] \to \mathbb{R}$ che risolve il Pb. di Cauchy.

Es : • $F(t,x) = Ax$, $A \in \mathbb{R}^{n \times n}$ (è continua!)

$$\Rightarrow \| Ax_1 - Ax_2 \| = \| A(x_1 - x_2) \| \le \underbrace{\| A \|}_{=L} \| x_1 - x_2 \|$$

è lipschitziana

• $F(t,x) = G(x) + f(t)$ con

$G : \mathbb{R}^n \to \mathbb{R}^n$ lipschitziana

$f : \mathbb{R} \to \mathbb{R}^n$ continua

è continua e lipschitz.

- $F(t,x) = f(x)$ con (di classe C^1)

$f: \mathbb{R} \to \mathbb{R}$ t.c. $\exists M : \forall x \in \mathbb{R},\ |f'(x)| \leq M$

$$\Rightarrow |F(t,x_1) - F(t,x_2)| = |f(x_1) - f(x_2)| =$$

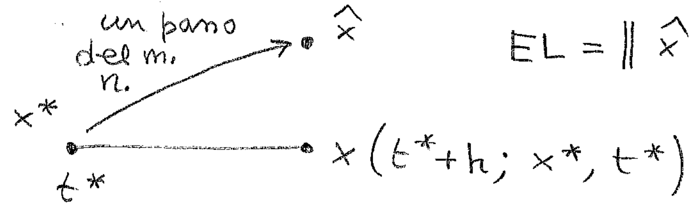
$$= |\underbrace{f'(\theta)}_{\theta \text{ tra } x_1 \text{ ed } x_2}(x_1 - x_2)| \leq \underbrace{M}_{=L}|x_1 - x_2|$$

è lipschitziana.

- Si consideri un metodo numerico ad un passo di <u>ordine</u> $p \geq 1$ ovvero t.c.:

$$\boxed{\forall\, t^*, x^* \quad \exists\, C(t^*,x^*) : \quad EL \leq C(t^*,x^*)\, h^{p+1}}$$

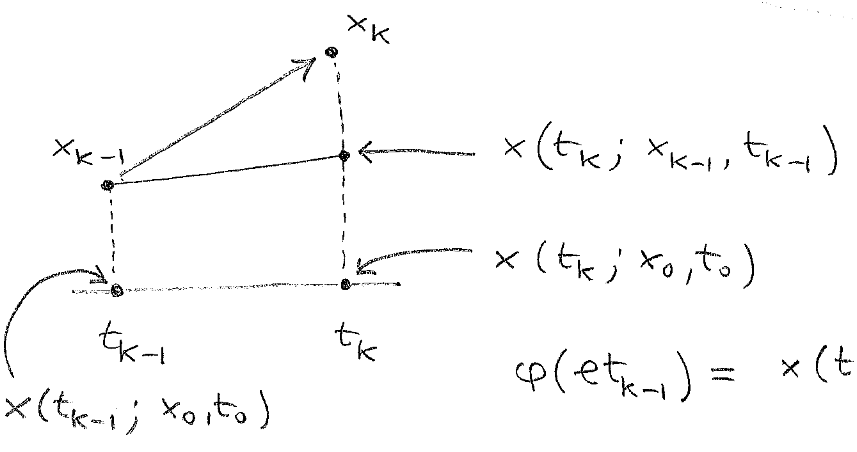
con:



$$EL = \| \hat{x} - x(t^*+h;\ x^*,t^*) \|$$

... a PASSO VARIABILE: a ciascuna iteraz il passo è scelto in modo che $EL \leq E$.

E: errore locale massimo consentito

- 

$$et_k = x_k - x(t_k;\ x_0,t_0)$$

$$el_k = x_k - x(t_k;\ x_{k-1},t_{k-1})$$

$$\varphi(et_{k-1}) = x(t_k;\ x_{k-1},t_{k-1}) - x(t_k;\ x_0,t_0)$$

$$\Rightarrow \quad et_k = el_k + \varphi\left(et_{k-1}\right)$$

errore totale all'ist t_k = errore locale all'ist t_k +

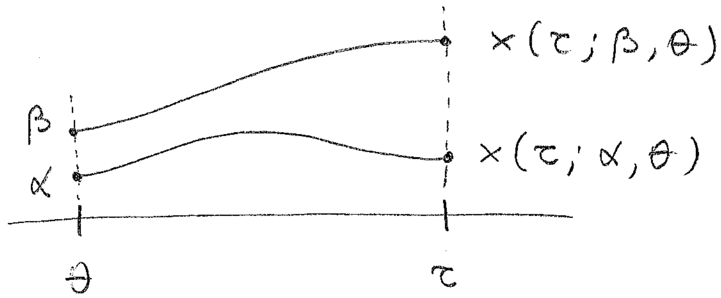
\+ come l'eq. diff. ha propagato l'errore et_{k-1} fino all'ist t_k

$$\varphi\left(et_{k-1}\right) = x\left(t_k ;\, x_{k-1},\, t_{k-1}\right) -$$

$$- x\left(t_k ;\, x\left(t_{k-1} ;\, x_0, t_0\right),\, t_{k-1}\right)$$

TEO: $\forall\ \theta < \tau \in [t_0, t_f]$, $\alpha, \beta \in \mathbb{R}^n$:

$$\| x(\tau; \alpha, \theta) - x(\tau; \beta, \theta) \| \leq e^{L|\tau - \theta|} \| \alpha - \beta \|$$



$$\Rightarrow \quad \| \varphi\left(et_{k-1}\right) \| \leq e^{L(t_k - t_{k-1})} \| et_{k-1} \|$$

Q. di:

$$\overset{ET_k}{\| et_k \|} = \| el_k + \varphi\left(et_{k-1}\right) \| \leq \overset{EL_k}{\| el_k \|} + \| \varphi\left(et_{k-1}\right) \|$$

$$\leq E + e^{L(t_k - t_{k-1})}\, ET_{k-1}$$

Ma:

$$ET_{k-1} = \| el_{k-1} + \varphi(et_{k-2}) \| \leq$$

$$\leq E + e^{L(t_{k-1} - t_{k-2})} ET_{k-2}$$

e:

$$ET_k \leq E + e^{L(t_k - t_{k-1})} \left[E + e^{L(t_{k-1} - t_{k-2})} ET_{k-2} \right]$$

$$= E \left[1 + e^{L(t_k - t_{k-1})} \right] + e^{L(t_k - t_{k-2})} ET_{k-2}$$

... procedendo all'indietro ...

$$ET_k \leq E \left[1 + e^{L(t_k - t_{k-1})} + e^{L(t_k - t_{k-2})} + \dots \right.$$

$$\left. \dots + e^{L(t_k - t_1)} \right] + e^{L(t_k - t_0)} ET_0$$

$\boxed{\text{ip: } ET_0 = 0}$; $\boxed{\forall k, j \text{ con } 1 \leq j \leq k : \quad e^{L(t_k - t_j)} \leq e^{L(t_k - t_0)}}$

$t_k - t_j \leq t_k - t_0$

dunque:

numero di passi: dip. da E!

$$\boxed{ET_k \leq k\, e^{L(t_k - t_0)} E \leq N e^{L(t_f - t_0)} E}$$

SE $\lim_{E \to 0} NE = 0$ allora: $\forall k, \lim_{E \to 0} ET_k = 0$

CONVERGENTE

- come cresce N quando $E \to 0$

Metodo di ordine p:

$$EL_k \leq C(t_{k-1}, x_{k-1})\, h_{k-1}^{p+1}$$

$$\boxed{\begin{array}{l}\exists\, K(t_0, x_0) \text{ t.c.} \\ \qquad \forall k,\ C(t_{k-1}, x_{k-1}) \leq K(t_0, x_0)\end{array}}$$

Scelta h_{k-1}: t.c. $C(t_{k-1}, x_{k-1})\, h_{k-1}^{p+1} = E$

$$\Rightarrow h_{k-1} = \left[\frac{E}{C(t_{k-1}, x_{k-1})}\right]^{\frac{1}{p+1}} \geq \left[\frac{1}{K(t_0, x_0)}\right]^{\frac{1}{p+1}} E^{\frac{1}{p+1}}$$

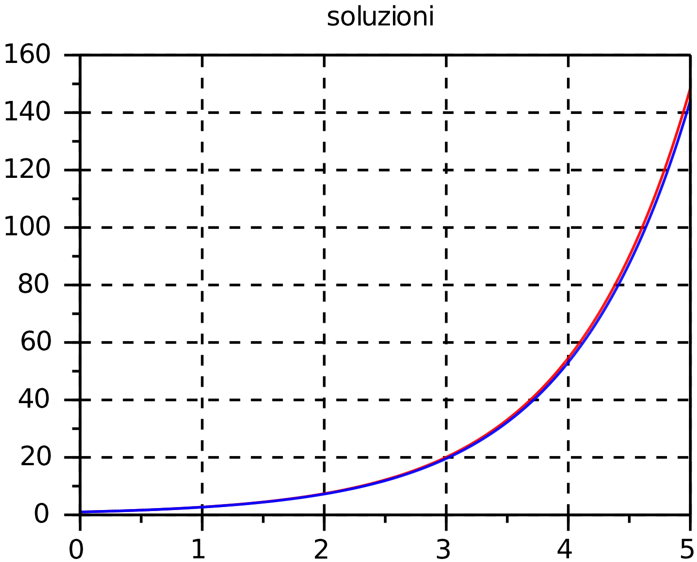
dunque:

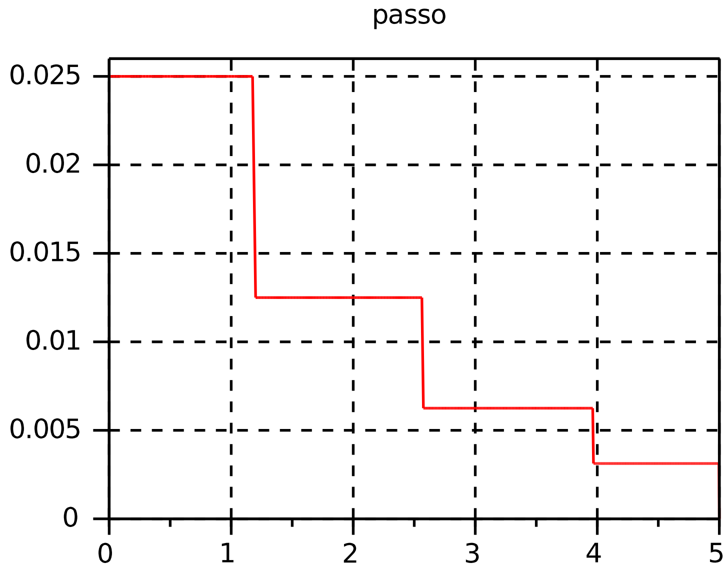
$$t_f - t_0 = h_0 + h_1 + \dots + h_{N-1} \geq N \left[\frac{1}{K(t_0, x_0)}\right]^{\frac{1}{p+1}} E^{\frac{1}{p+1}}$$

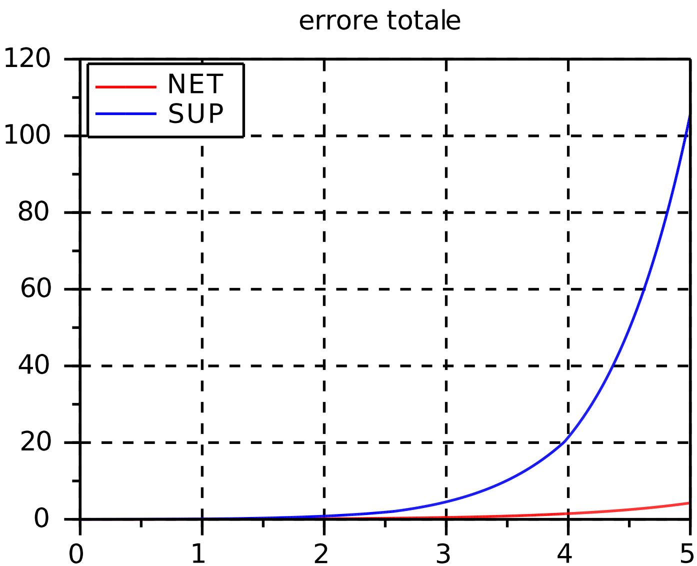
$$\Rightarrow \boxed{N \leq (t_f - t_0) \frac{(K(t_0, x_0))^{\frac{1}{p+1}}}{E^{\frac{1}{p+1}}}}$$

Infine: $\boxed{NE \leq (t_f - t_0)\, K(t_0, x_0)^{\frac{1}{p+1}}\, E^{\frac{p}{p+1}}}$

e $\boxed{ET_k \leq (t_f - t_0)\, e^{L(t_f - t_0)}\, K(t_0, x_0)^{\frac{1}{p+1}}\, E^{\frac{p}{p+1}}}$







equazione: dx/dt = L x, x(0) = 1 con L =1.000D+00

EL_MAX = 1.000D-03

Errore totale massimo = 4.274D+00

Numero passi = 7.120D+02

Passo:

...minimo = 3.375D-14

...massimo = 2.500D-02

...medio = 7.022D-03