* CONDIZIONAMENTO *

dati: $A \in \mathbb{R}^{n\times n}$ invert, $b \in \mathbb{R}^n$

soluz: x^* t.c. $Ax^* = b$

dati perturbati: $A + \delta A \in \mathbb{R}^{n\times n}$ invert, $b + \delta b \in \mathbb{R}^n$

PERTURBAZ (additiva) dei dati

soluzione: $\hat{x}$ t.c. $(A+\delta A)\hat{x} = b + \delta b$

Es: $A = \begin{bmatrix} 10 & 6 \\ -1 & 4 \end{bmatrix}$, $A + \delta A = \begin{bmatrix} 9 & 6{,}1 \\ -1 & 3{,}8 \end{bmatrix} \Rightarrow \delta A = \begin{bmatrix} -1 & 0{,}1 \\ 0 & -0{,}2 \end{bmatrix}$

Allora: $\hat{x} - x^* = (A+\delta A)^{-1}(b+\delta b) - A^{-1}b = F(A, b; \delta A, \delta b)$

δx: PERTURBAZ delle soluz

Pb: studiare F

- CASO I: $\delta A = 0$, $b \neq 0$

$\hat{x} = A^{-1}(b+\delta b)$, $\delta x = \hat{x} - x = A^{-1}\delta b$

$\|\delta x\| = \|A^{-1}\delta b\|$; $\epsilon_d = \dfrac{\|\delta x\|}{\|x^*\|}$ ERRORE RELATIVO

$$\epsilon_d = \frac{\|A^{-1}\delta b\|}{\|A^{-1}b\|} \underset{\exists\, \delta b \text{ t.c. } =}{\le} \|A^{-1}\| \frac{\|\delta b\|}{\|b\|} \frac{\|b\|}{\|A^{-1}b\|} \underset{\exists\, b \text{ t.c. } =}{\le} \|A^{-1}\| \, \|A\| \underbrace{\frac{\|\delta b\|}{\|b\|}}_{\epsilon_b}$$

$$\left(\text{oss: } \left\{ \frac{\|Av\|}{\|v\|},\ v \neq 0 \right\} = \left\{ \frac{\|w\|}{\|A^{-1}w\|},\ w \neq 0 \right\} \right)$$

def (numero di condiz): $A \in \mathbb{R}^{n\times n}$, invert

$$c(A) = \|A\| \, \|A^{-1}\|$$

NUMERO DI CONDIZIONAMENTO di A (in norma ...)

Teo (condiz, I): $A \in \mathbb{R}^{n\times n}$ invert;

- $\forall b \in \mathbb{R}^n, \neq 0$, $\forall \delta b \in \mathbb{R}^n$: $\epsilon_d \le c(A)\,\epsilon_b$
- $\exists b \in \mathbb{R}^n, \neq 0$, $\exists \delta b \in \mathbb{R}^n$: $\epsilon_d = c(A)\,\epsilon_b$

Es: $A = \begin{bmatrix} 10^{-3} & 0 \\ 0 & 10^3 \end{bmatrix}$; $\det A = 1$, $c_\infty(A) = 10^6$ ($\gg 1$!)

- $x^* = \begin{bmatrix} 10^3 b_1 \\ 10^{-3} b_2 \end{bmatrix}$, $\hat{x} = x^* + \delta x = \begin{bmatrix} 10^3(b_1 + \delta b_1) \\ 10^{-3}(b_2 + \delta b_2) \end{bmatrix}$, $\delta x = \begin{bmatrix} 10^3 \delta b_1 \\ 10^{-3} \delta b_2 \end{bmatrix}$

(1) $b = \binom{1}{0} \Rightarrow x^* = \binom{10^3}{0}$, $\|x^*\|_\infty = 10^3$

- $\epsilon_b = \dfrac{\|\delta b\|_\infty}{\|b\|_\infty} = \|\delta b\|_\infty$, $\|\delta x\|_\infty \le 10^3 \|\delta b\|_\infty$

$$\Rightarrow \epsilon_d = \frac{\|\delta x\|_\infty}{\|x^*\|_\infty} \le \frac{10^3 \|\delta b\|_\infty}{10^3} = \|\delta b\|_\infty = \epsilon_b$$

- Teo condiz, I $\Rightarrow \epsilon_d \le 10^6 \epsilon_b$: molto pessimista!

(2) $b = \binom{0}{1}$, $\delta b = \binom{\delta b_1}{0} \Rightarrow x^* = \binom{0}{10^{-3}}$, $\|x^*\|_\infty = 10^{-3}$,

$\delta x = \binom{10^3 \delta b_1}{0}$

- $\epsilon_b = \|\delta b\|_\infty$, $\|\delta x\|_\infty = 10^3 \|\delta b\|_\infty$

$$\Rightarrow \quad \epsilon_d = \frac{\|\delta x\|_\infty}{\|x^*\|_\infty} \doteq 10^3 \frac{\|\delta b\|_\infty}{10^{-3}} = 10^6 \, \|\delta b\|_\infty = 10^6 \, \epsilon_b$$

- Teo condiz, I non ferimiento !

- CASO 2 : $\delta b = 0$, δA t.c. $A + \delta A$ invert

$$\epsilon_A = \frac{\|\delta A\|}{\|A\|}, \quad \hat{\epsilon}_d = \frac{\|\delta x\|}{\|\hat{x}\|}$$

> TEO (condiz, II) : $A \in \mathbb{R}^{n \times n}$, invert ;
>
> - $\forall b \in \mathbb{R}^n, \neq 0$, $\forall \delta A \in \mathbb{R}^{n \times n}$ t.c. $A + \delta A$ invert : $\hat{\epsilon}_d \leq c(A)\, \epsilon_A$
> - $\exists b \in \mathbb{R}^n, \neq 0$, $\exists \delta A \in \mathbb{R}^{n \times n}$ t.c. $A + \delta A$ invert : $\hat{\epsilon}_d = c(A)\, \epsilon_A$

Oss : $\mathbb{R}^n$, N ; $A \in \mathbb{R}^{n \times n}$ invert : $c(A) \geq 1$

(dim : $I = A^{-1} A \Rightarrow 1 = \|I\| \leq \|A^{-1}\| \, \|A\|$)