- Norme di matrici

def (norma di matrice):

Si cons $\mathbb{R}^n$ con norma $N$; $A \in \mathbb{R}^{n\times n}$

$$\|A\|_N = \sup\left\{ \frac{N(Av)}{N(v)},\ v \neq 0 \right\}$$ "norma di A INDOTTA da N"

Es: $\mathbb{R}^n, N$; $\|I\|_N = 1$

Os: $\sup\{\#\} < +\infty$

Es: $N_\infty(Av) \leq [N_\infty(a_1) + \dots]\, N_\infty(v)$

Os (formule di calcolo):

$$N_1(A) = \max\{N_1(a_1), \dots, N_1(a_n)\} = \|A\|_1$$

$$N_\infty(A) = \max\{N_1(r_1^T), \dots, N_1(r_n^T)\} = \|A\|_\infty \quad (= \|A^T\|_1)$$

$$N_2(A) = \sqrt{\max\{\lambda \in \mathbb{R} \mid \lambda \text{ autovalore di } A^TA\}}$$

Os: $A^TA$ è semidef pos $\Rightarrow$ autovalori $\begin{cases} \in \mathbb{R} \\ \geq 0 \end{cases}$

Es: $A = \begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix}$

- $\|A\|_1 = \max\{3, 2\} = 3$
- $\|A\|_\infty = \max\{4, 1\} = 4$
- $A^TA = \begin{pmatrix} 9 & -3 \\ -3 & 2 \end{pmatrix}$, $\|A\|_2 = \sqrt{\max\{\lambda_1, \lambda_2\}}$

- Proprietà delle NORME INDOTTE: $\mathbb{R}^n, N$

(I) $\forall A \in \mathbb{R}^{n\times n}, v \in \mathbb{R}^n$: $N(Av) \leq \|A\|_N\, N(v)$

Es (per casa): dimostrarlo usando la def di norma indotta

(II) $\forall A, B \in \mathbb{R}^{n\times n}$, $\|AB\|_N \leq \|A\|_N \|B\|_N$

(dim: no)

(III) $\forall A \in \mathbb{R}^{n\times n}$: $\|A\|_N = \max\{N(Av),\ N(v) = 1\}$

$\forall A \in \mathbb{R}^{n\times n}$ invert: $\|A^{-1}\| = \left(\min\{N(Av),\ N(v) = 1\}\right)^{-1}$

Es:

- $A \in \mathbb{R}^{n\times n}$ invertibile;

$A^{-1}A = I \Rightarrow \|A^{-1}\|_N \geq \|A\|_N^{-1}$

- $A \in \mathbb{R}^{n\times n}$ invert, $b \in \mathbb{R}^n$, $x^* \in \mathbb{R}^n$ t.c. $Ax^* = b$

$$\Rightarrow \frac{\|b\|_N}{\|A\|_N} \leq \|x^*\|_N \leq \|A^{-1}\|_N \|b\|_N$$