$x^* \mid Ax^* = b$ soluz del sist non perturbato

$\hat{x} \mid (A+\delta A)\hat{x} = b+\delta b$ soluz del sist perturbato

PERTURBAZ dei dati

$\delta x = \hat{x} - x^* = F(A, b; \delta A, \delta b)$

• CASO 1 : $\delta A = 0$, $b \neq 0$

$\hat{x} = A^{-1}(b+\delta b)$; $\delta x = A^{-1}\delta b$

$\|\delta x\| = \|A^{-1}\delta b\|$; $\boxed{\varepsilon_d = \frac{\|\delta x\|}{\|x^*\|}}$ ERRORE RELATIVO

$$\varepsilon_d = \frac{\|A^{-1}\delta b\|}{\|A^{-1}b\|} \le \|A^{-1}\| \frac{\|\delta b\|}{\|b\|} \frac{\|b\|}{\|A^{-1}b\|} \le \|A^{-1}\| \, \|A\| \frac{\|\delta b\|}{\|b\|}$$

$\exists\, \delta b$ t.c. $=$ (dalla def di $\|A^{-1}\|$)

$$\boxed{\varepsilon_b = \frac{\|\delta b\|}{\|b\|}}$$

$\exists\, b$ t.c. $=$

Oss: $\inf\left\{\frac{\|A^{-1}x\|}{\|x\|},\ x \neq 0\right\} \le \frac{\|A^{-1}b\|}{\|b\|}$

$\quad = \min\left\{\|A^{-1}x\|,\ \|x\|=1\right\}$

perciò: $\frac{\|b\|}{\|A^{-1}b\|} \le \left(\min\left\{\|A^{-1}x\|, \|x\|=1\right\}\right)^{-1} = \|A\|$

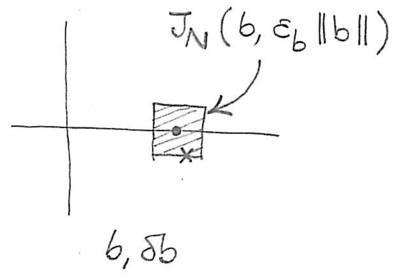
def (numero di condiz): $A \in \mathbb{R}^{n\times n}$, invertibile;

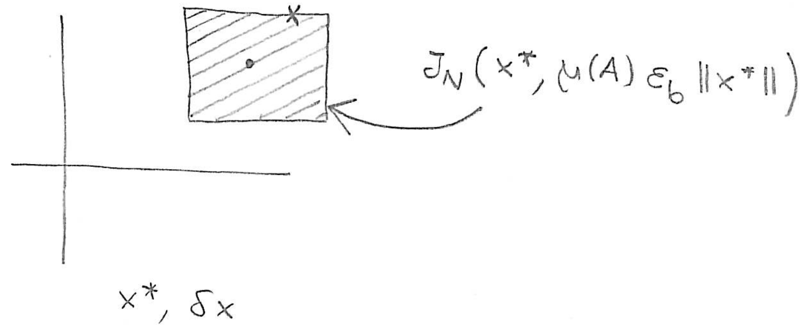
$\mu(A) = \|A\| \, \|A^{-1}\|$ NUMERO di CONDIZIONAM di A

TEO (condiz, I): $A \in \mathbb{R}^{n\times n}$, invert;

- $\forall b \in \mathbb{R}^n, \neq 0$, $\forall \delta b \in \mathbb{R}^n$: $\varepsilon_d \le \mu(A)\, \varepsilon_b$
- $\exists b \in \mathbb{R}^n, \neq 0$, $\exists \delta b \in \mathbb{R}^n$: $\varepsilon_d = \mu(A)\, \varepsilon_b$

Oss:





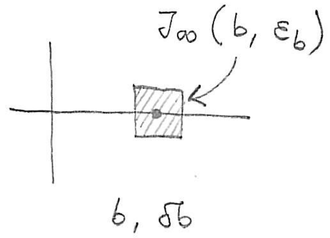
Es: $A = \begin{bmatrix} 10^{-3} & 0 \\ 0 & 10^3 \end{bmatrix}$ • $\det A = 1$, $\mu_\infty(A) = 10^6$

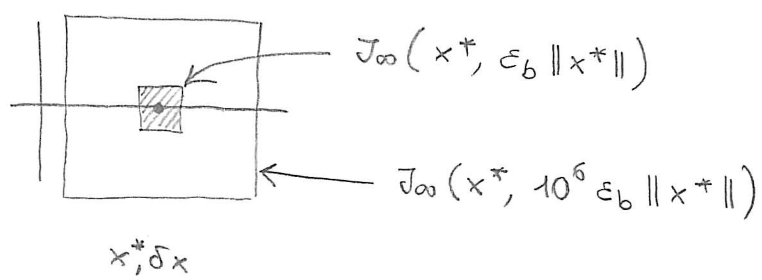
• $x^* = \begin{bmatrix} 10^3 b_1 \\ 10^{-3} b_2 \end{bmatrix}$, $\hat{x} = x^* + \delta x = \begin{bmatrix} 10^3 (b_1+\delta b_1) \\ 10^{-3}(b_2+\delta b_2) \end{bmatrix}$, $\delta x = \begin{bmatrix} 10^3 \delta b_1 \\ 10^{-3}\delta b_2 \end{bmatrix}$

(1) $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow x^* = \begin{pmatrix} 10^3 \\ 0 \end{pmatrix}$, $\|x^*\|_\infty = 10^3$

• $\varepsilon_b = \frac{\|\delta b\|_\infty}{\|b\|_\infty} = \|\delta b\|_\infty$

• $\|\delta x\|_\infty \le 10^3 \|\delta b\|_\infty \Rightarrow \boxed{\varepsilon_d \le \frac{10^3 \|\delta b\|_\infty}{10^3} = \|\delta b\|_\infty = \varepsilon_b}$



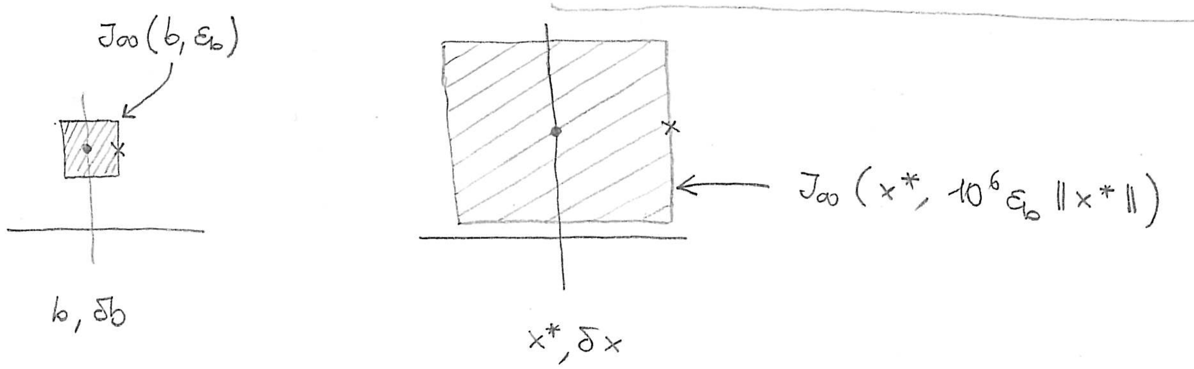


(2) $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\delta b = \begin{pmatrix} \delta b_1 \\ 0 \end{pmatrix} \Rightarrow x^* = \begin{pmatrix} 0 \\ 10^{-3} \end{pmatrix}$, $\|x^*\|_\infty = 10^{-3}$,

$\delta x = \begin{pmatrix} 10^3 \delta b_1 \\ 0 \end{pmatrix}$

- $\varepsilon_b = \| \delta b \|_\infty$

- $\| \delta x \|_\infty = 10^3 \| \delta b \|_\infty \Rightarrow \boxed{\varepsilon_d = \dfrac{10^3 \| \delta b \|_\infty}{10^{-3}} = 10^6 \| \delta b \|_\infty = 10^6 \varepsilon_b}$



- CASO 2 : $\delta b = 0$, δA t.c. $A + \delta A$ invert

$\boxed{\varepsilon_A = \dfrac{\|\delta A\|}{\|A\|}}$ $\boxed{\hat{\varepsilon}_d = \dfrac{\|\delta x\|}{\|\hat{x}\|}}$

TEO (condiz. II) : $A \in \mathbb{R}^{n \times n}$, invert ;

- $\forall b \begin{cases} \in \mathbb{R}^n \\ \neq 0 \end{cases}$, $\forall \delta A \begin{cases} \in \mathbb{R}^{n \times n} \\ \text{t.c. } A + \delta A \text{ invert} \end{cases}$: $\hat{\varepsilon}_d \leq \mu(A)\, \varepsilon_A$

- $\exists b \begin{cases} \in \mathbb{R}^n \\ \neq 0 \end{cases}$, $\exists \delta A \begin{cases} \in \mathbb{R}^{n \times n} \\ \text{t.c. } A + \delta A \text{ invert} \end{cases}$: $\hat{\varepsilon}_d = \mu(A)\, \varepsilon_A$

Oss: $(\mathbb{R}^n, N)$, $A \in \mathbb{R}^{n \times n}$ invert : $\mu(A) \geq 1$

(dim : $I = A^{-1} A \Rightarrow 1 = \|I\|_N \leq \|A^{-1}\|_N \|A\|_N$)

Es : $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$; $b, \delta A$ t.c. $\hat{x} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$, $\varepsilon_A \leq \dfrac{1}{100}$

$\downarrow$

$\dfrac{\|\delta A\|_2}{\|A\|_2}$

- $\|A\|_2 = \sqrt{\max\{1, 9\}} = 3$

$p_A(x) = (2-x)^2 - 1 = (2 - x - 1)(2 - x + 1)$

$= (1 - x)(3 - x)$, $\lambda_1 = 1$, $\lambda_2 = 3$

$A^T A = A^2$ (perché A è simm!)

$\Rightarrow$ autovalori di $A^T A = \lambda_1^2, \lambda_2^2$

- $\|A^{-1}\|_2 = \sqrt{\max\{1, 1/9\}} = 1$

$\boxed{\text{Oss : } A \text{ simm} \Rightarrow \mu_2(A) = \dfrac{\max |\lambda_k|}{\min |\lambda_k|}}$

- $\mu_2(A) = 3 \Rightarrow \hat{\varepsilon}_d \leq \mu_2(A)\, \varepsilon_A \leq \dfrac{3}{100}$

$\Rightarrow \|\delta x\|_2 = \hat{\varepsilon}_d \|\hat{x}\|_2 \leq \dfrac{3}{100} \cdot 10 = \dfrac{3}{10}$