Es: $A = \begin{bmatrix} 2 & -1 & -2 & 2 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathbb{C}^{4\times 4}$; determ $FCJ(A)$.

- $p_A(x) = (1-x)^3(-x) \Rightarrow \exists\ J_\#(0),\ J_\#(1)$ e $\Sigma \dim(0) = 1$, $\Sigma \dim(1) = 3$
- $\dim \ker A = 1$, $\dim \ker(A-I) = 2 \Rightarrow J_1(0),\ J_1(1),\ J_2(1)$

q. di: $\boxed{FCJ(A) = \mathrm{diag}(J_1(0), J_1(1), J_2(1))}$

Es: $A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix} \in \mathbb{C}^{3\times 3}$; determ $FCJ(A)$

(Soluz: $FCJ(A) = \mathrm{diag}(J_1(1), J_2(2))$)

Es: $A = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \in \mathbb{C}^{4\times 4}$; determ $FCJ(A)$

- $p_A(x) = (2-x)^4 \Rightarrow J_\#(2)$ e $\Sigma \dim(2) = 4$
- $\dim \ker(A-2I) = 2 \Rightarrow$ 2 blocchi ..
- $\dim \ker(A-2I)^2 = 3 \Rightarrow J_1(2),\ J_3(2)$

qdi: $\boxed{FCJ(A) = \mathrm{diag}(J_1(2), J_3(2))}$

Oss:
- due elem di $\mathbb{C}^{n\times n}$ in f di J ottenuti uno dall'altro permutando i blocchi sulla diagonale sono simili
- data $A \in \mathbb{C}^{n\times n}$, tutte le matrici in f di J simili ad A hanno sulla diagonale gli stessi blocchi ma in ordine diverso, dunque sono tutte simili.

Es: Per ciascuno degli Es precedenti, determ tutte le matr in f di J simili ad A.

* DETERMINAZIONE di UNA MATR CHE REALIZZA LA SIMILITUDINE *

Pb: data A e $FCJ(A)$, determ C invert t.c.

$$AC = C\, FCJ(A)$$

Es: $A = \begin{bmatrix} 2 & -1 & -2 & 2 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathbb{C}^{4\times 4}$, $FCJ(A) = \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}$

Cerco $c_1, \dots, c_4 \in \mathbb{C}^4$ t.c. posto $C = (c_1, c_2, c_3, c_4)$ si abbia:

(I) C invertibile (ovvero $c_1, \dots, c_4$ lin indip)

(II) $AC = C\, FCJ(A)$

(II) $\Rightarrow$
$Ac_1 = 0 \Rightarrow c_1$ elem non nullo di Ker A
$Ac_2 = c_2 \Rightarrow c_2$ elem non nullo di $\mathrm{Ker}(A-I)$
$Ac_3 = c_3 \Rightarrow c_3$ " " " " "
$Ac_4 = c_3 + c_4 \Leftrightarrow (A-I)c_4 = c_3$

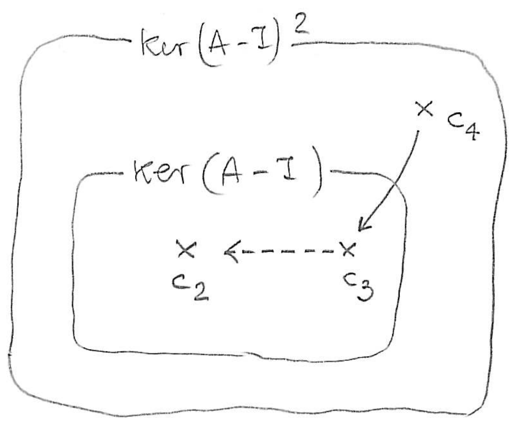
$c_3 \neq 0 \Rightarrow c_4 \notin \mathrm{Ker}(A-I)$

MA $(A-I)^2 c_4 = (A-I)c_3 = 0$, ovvero c_4 elem di $\mathrm{Ker}(A-I)^2$

Poiché...
- $\dim \mathrm{Ker}\, A = 1$
- $\dim \mathrm{Ker}(A-I) = 2$
- $\dim \mathrm{Ker}(A-I)^2 = 3$ e $\mathrm{Ker}(A-I) \subset \mathrm{Ker}(A-I)^2$

procedura per det $c_1, \dots, c_4$:

(1) c_1 = un quals elem non nullo di Ker A



(2) c_4 = un quals elem di $\ker(A-I)^2 \setminus \ker(A-I)$

(3) $c_3 = (A-I)\,c_4$

(4) c_2 un quals elem di $\ker(A-I)$ lin indip da c_3 (esiste certam perché $\dim \ker(A-I) = 2$)

Si ha:

- $\ker A = \left\langle \begin{bmatrix}1\\0\\1\\0\end{bmatrix} \right\rangle$
- $\ker(A-I) = \left\langle \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix} \right\rangle$, $\ker(A-I)^2 = \left\langle \overbrace{\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}}, \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \right\rangle$

Allora: $c_1 = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}$, $c_4 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ scelti

$c_3 = (A-I)\,c_4 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}$ calcolato

$c_2 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$ scelto lin indip da c_3

e $C = \begin{bmatrix}1&0&1&1\\0&0&1&0\\1&1&0&0\\0&1&0&0\end{bmatrix}$ (che ris invertibile!)

Es: $A = \begin{bmatrix}2&-1&-2&3\\1&0&-1&1\\0&0&0&1\\0&0&0&1\end{bmatrix} \in \mathbb{C}^{4\times 4}$; determ FCJ(A) e matr che realizza la sim

- $p_A(x) = (-x)(1-x)^3 \Rightarrow J_1(0),\ J_\#(1)$ e $\Sigma \dim(1) = 3$
- $\dim \ker(A-I) = 1 \Rightarrow J_3(1)$

quindi $\boxed{FCJ(A) = \mathrm{diag}(J_1(0), J_3(1))}$

- cerco $c_1, \dots, c_4 \in \mathbb{C}^4$ ...
- $AC = C\, FCJ(A)$ ovvero

$Ac_1 = 0 \Rightarrow c_1$ elem non nullo di $\ker A$

$Ac_2 = c_2 \Rightarrow c_2$ elem non nullo di $\ker(A-I)$

$Ac_3 = c_2 + c_3 \sim (A-I)c_3 = c_2 \Rightarrow c_3 \in \ker(A-I)^2 \setminus \ker(A-I)$

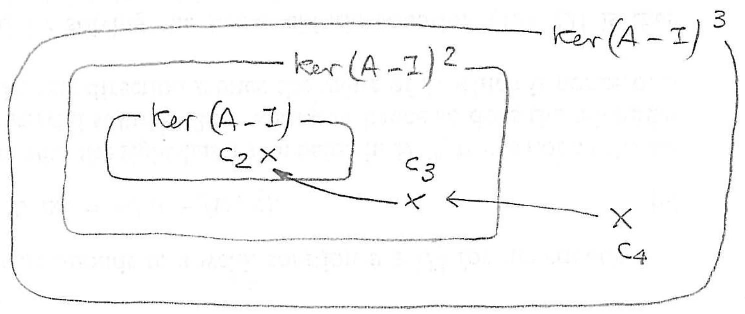
$Ac_4 = c_3 + c_4 \sim (A-I)c_4 = c_3 \Rightarrow c_4 \in \ker(A-I)^3 \setminus \ker(A-I)^2$

- $\dim \ker A = 1$

$\dim \ker(A-I) = 1$

$\dim \ker(A-I)^2 = 2$

$\dim \ker(A-I)^3 = 3$



- procedura:

(1) c_1 un quals elem non nullo di $\ker A$

(2) $c_4 \in \ker(A-I)^3 \setminus \ker(A-I)^2$

(3) $c_3 = (A-I)\,c_4$

(4) $c_2 = (A-I)\,c_3$

- $\ker A = \left\langle \begin{bmatrix}1\\0\\1\\0\end{bmatrix} \right\rangle$

$\ker(A-I) = \left\langle \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \right\rangle$, $\ker(A-I)^2 = \left\langle \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \right\rangle$,

$\ker(A-I)^3 = \left\langle \underbrace{\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\end{bmatrix}}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix} \right\rangle$

- $c_1 = \begin{pmatrix}1\\0\\1\\0\end{pmatrix}$, $c_4 = \begin{pmatrix}0\\0\\1\\1\end{pmatrix}$ scelti

$c_3 = (A-I)\,c_4 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$, $c_2 = (A-I)\,c_3 = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}$ calcolati e $\quad C = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (che risulta invertibile!).

Es: $A = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ -1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 3 \end{bmatrix} \in \mathbb{C}^{4\times 4}$; determ FCJ(A) e matr che realizza la sim

- $p_A(x) = (3-x)^4$
- $\dim \ker (A-3I) = 2$, $\dim \ker (A-3I)^2 = 4$

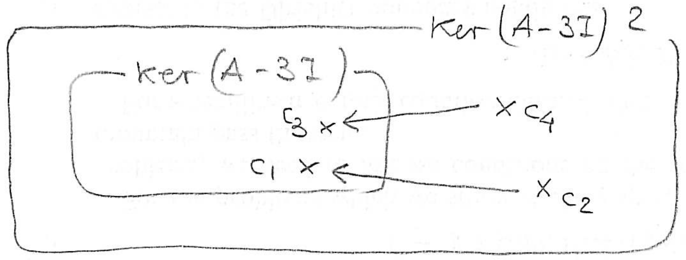
$\Rightarrow$ FCJ(A) $=$ diag$(J_2(3), J_2(3))$

- $c_1, \dots, c_4 \dots \quad AC = C\,$FCJ(A)
- $Ac_1 = 3c_1 \Rightarrow c_1$ elem non nullo di $\ker (A-3I)$

  $Ac_2 = c_1 + 3c_2 \Rightarrow c_2 \in \ker (A-3I)^2 \setminus \ker (A-3I)$

  $Ac_3 = 3c_3 \Rightarrow c_3$ elem non nullo di $\ker (A-3I)$

  $Ac_4 = c_3 + 3c_4 \Rightarrow c_4 \in \ker (A-3I)^2 \setminus \ker (A-3I)$
- $\dim \ker (A-3I) = 2$, $\dim \ker (A-3I)^2 = 4$



- procedura:

  (1) $c_2 \in \ker (A-3I)^2 \setminus \ker (A-3I)$

  (2) $c_1 = (A-I)\,c_2$

  (3) $c_4 \in \ker (A-3I)^2 \setminus \ker (A-3I)$ lin indip da c_2

  (4) $c_3 = (A-I)\,c_4$

- $\ker (A-3I) = \left\langle \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle$

  $\ker (A-3I)^2 = \left\langle \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle$

- $c_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $c_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

  $c_1 = (A-I)\,c_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$, $\quad c_3 = (A-I)\,c_4 = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow \quad C = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ (è invertibile!)

Oss: Se $A \in \mathbb{C}^{n\times n}$ è ad elementi reali

e tutti gli autovalori di A sono reali

$\Rightarrow$ FCJ(A) è ad elem reali

e $\exists\ C \in \mathbb{C}^{n\times n}$ ad elem reali che realizza

la similitudine.
