- $\mathcal{S}: \begin{cases} 3x_1 - x_2 + x_4 = 3 \\ x_2 + x_3 = 2 \\ x_1 - x_4 = 0 \end{cases}, \quad x \in \mathbb{R}^4$

- $A = \begin{bmatrix} 3 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \in \mathbb{R}^{3\times 4}$ matr <u>incompleta</u> di $\mathcal{S}$

  $b = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \in \mathbb{R}^3$ colonna dei t. noti di $\mathcal{S}$

- $(A,b) \in \mathbb{R}^{3\times 5}$ matr <u>completa</u> di $\mathcal{S}$

- $\mathcal{S}$ equiv a: $Ax = b$, $x \in \mathbb{R}^4$; $x_1 a_1 + \dots + x_4 a_4 = b$, $x \in \mathbb{R}^4$; $\begin{cases} a_1' x = b_1 \\ \vdots \\ a_4' x = b_4 \end{cases}$, $x \in \mathbb{R}^4$

- $S(A,b) \subset \mathbb{R}^4$ ins soluzioni: $\neq \emptyset$: $\mathcal{S}$ <u>risolubile</u>; $= \emptyset$: $\mathcal{S}$ <u>non risolubile</u>

- $Ax = 0$ sist <u>omogeneo associato</u> al $\mathcal{S}$;

  $S(A,0)$ è ssv di $\mathbb{R}^4$

<u>TEO</u> (Rouché - Capelli)

$Ax = b$ risolubile $(S(A,b) \neq \emptyset) \iff \mathrm{rk}(A,b) = \mathrm{rk}(A)$

<u>dim</u>: $\mathrm{rk}(A,b) = \mathrm{rk}(A) \iff \dim\langle a_1,\dots,a_4, b\rangle = \dim\langle a_1,\dots,a_4\rangle$

$\iff b \in \langle a_1,\dots,a_4\rangle \iff \exists\, x \in \mathbb{R}^4$ t.c. $Ax = b$

- $\mathcal{S}$ risolubile e $\tilde{x} \in S(A,b)$ <u>allora</u>

$$\boxed{S(A,b) = \tilde{x} + S(A,0)}$$

<u>dim</u>: ① $\tilde{x} + S(A,0) \subset S(A,b)$:

$y \in S(A,0) \Rightarrow A(\tilde{x}+y) = A\tilde{x} + Ay = b + 0 = b$

$\Rightarrow \tilde{x} + y \in S(A,b)$

② $S(A,b) \subset \tilde{x} + S(A,0)$

$x \in S(A,b)$, $z = x - \tilde{x} \Rightarrow Az = A(x - \tilde{x}) =$

$= Ax - A\tilde{x} = b - b = 0 \Rightarrow z \in S(A,0)$

ovvero: $x = \tilde{x} + z$ con $z \in S(A,0)$

cioè $x \in \tilde{x} + S(A,0)$.

- $\mathrm{rk}(A) = 3$ e a_1, a_2, a_3 base di $\langle a_1,\dots,a_4\rangle$;

① $\forall t \in \mathbb{R}$, $\exists!\, x_1, x_2, x_3 \in \mathbb{R}$ t.c. $x_1 a_1 + x_2 a_2 + x_3 a_3 = -t a_4$

ovvero: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ t \end{bmatrix} \in S(A,0)$

② $t=1$, $\hat{x}_1, \hat{x}_2, \hat{x}_3 \in \mathbb{R}$ t.c. $\hat{x}_1 a_1 + \hat{x}_2 a_2 + \hat{x}_3 a_3 = -a_4$

allora: $\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ 1 \end{bmatrix}$ è base di $S(A,0)$.

$\left(\xi \in S(A,0) \Rightarrow \xi = \xi_4 \begin{pmatrix} \hat{x}_1 \\ \vdots \\ 1 \end{pmatrix} \dots \right)$

dunque: $\boxed{\dim S(A,0) = 1 = \underbrace{4}_{\#\text{ incognite}} - \operatorname{rk}(A)}$

Es: $\mathcal{Y} = \begin{cases} x_1 = 1 \\ x_2 = 1 \\ x_1 = 1 \end{cases}, \quad x \in \mathbb{R}^2$

- $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$, $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
- $\operatorname{rk}(A,b) = 2$, $\operatorname{rk}(A) = 2$

  Teo R-C $\Rightarrow$ $\mathcal{Y}$ risolubile.
- $S(A,0) = \{0\}$ perché $x_1 a_1 + x_2 a_2 = 0 \Leftrightarrow x_1 = x_2 = 0$
- $\tilde{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in S(A,b)$ (verificare!)

  Allora: $S(A,b) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \{0\} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Oss: $\dim S(A,0) = 2 - 2 = 0$.

Es: $\mathcal{Y}: \begin{cases} x_1 + x_3 = 1 \\ x_1 + x_2 = 0 \\ x_2 = 0 \end{cases}, \quad x \in \mathbb{R}^4$

- $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 4}$, $b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^3$
- $\operatorname{rk}(A,b) = 3$, $\operatorname{rk}(A) = 3 \Rightarrow$ risolubile
- $S(A,0) = \left\langle \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$

  $\dim S(A,0) = 4 - 3 = 1$
- $\tilde{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \in S(A,b) \Rightarrow S(A,b) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$

Es: $\mathcal{Y}: \begin{cases} x_1 + x_2 = 0 \\ x_1 - x_2 = 1 \end{cases}, \quad x \in \mathbb{R}^4$

- $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 4}$, $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- $\operatorname{rk}(A) = 2$, $\operatorname{rk}(A,b) = 2 \Rightarrow$ risolubile
- $S(A,0) = \left\langle \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle$, $\dim S(A,0) = 4 - 2 = 2$
- $\tilde{x} = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \in S(A,b)$

  Allora: $S(A,b) = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$