

* DETERMINANTE *

def (determinante):

K campo,

$$F: K^{n \times n} \rightarrow K$$

t.c. $\forall A = (a_1, \dots, a_n) \in K^{n \times n}$:

[D1] $\forall t \in K, \forall j \in \{1, \dots, n\}: F(a_1, \dots, t a_j, \dots, a_n) = t F(a_1, \dots, a_j, \dots, a_n)$

[D2] $\forall j \in \{1, \dots, n\}, b \in K^n:$

$$F(a_1, \dots, a_j + b, \dots, a_n) = F(a_1, \dots, a_j, \dots, a_n) + F(a_1, \dots, b, \dots, a_n)$$

[D3] se $\exists i \neq j \in \{1, \dots, n\}$ t.c. $a_i = a_j$ allora $F(a_1, \dots, a_n) = 0$

[D4] $F(e_1, \dots, e_n) = 1$

TEO: $\forall n$ intero ≥ 1 , $\exists!$ $F: K^{n \times n} \rightarrow K$ che verifica

[D1], ..., [D4].

Tale funz si dice DETERMINANTE e si indica con det.

ES: $\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$; $\det \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \det(2e_1, e_2) = 2 \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2$;
 $\det \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \det(e_1, e_1 + e_2) = \det \begin{pmatrix} e_1 & e_2 \\ \parallel & \parallel \\ 1 & 0 \end{pmatrix} + \det \begin{pmatrix} e_1 & e_1 \\ \parallel & \parallel \\ 0 & 1 \end{pmatrix} = 1$

Oss: Sia $A = (a_1, \dots, a_n) \in K^{n \times n}$

- 1) SE $\exists j$ t.c. $a_j = 0$ allora $\det A = 0$ ([D1]...)
- 2) $\forall i \neq j: \det(a_1, \dots, a_i, \dots, a_j, \dots, a_n) = - \det(a_1, \dots, a_j, \dots, a_i, \dots, a_n)$
(scambio righe \Rightarrow cambio segno)

ES: $\det(a_1 + a_2, a_1 + a_2) = 0$
 \parallel [D2]...

$$\det(a_1, a_1) + \det(a_2, a_1) + \det(a_1, a_2) + \det(a_2, a_2)$$

\parallel [D3] \parallel [D3]
 0 0

$$\Rightarrow \det(a_2, a_1) = - \det(a_1, a_2)$$

(3) SE a_1, \dots, a_n sono lin. dep allora $\det A = 0$.

ES: $a_1 = a_2 + 3a_3 \Rightarrow \det(a_1, a_2, a_3) = \det(a_2 + 3a_3, a_2, a_3)$
 $= \det(a_2, a_2, a_3) + 3 \det(a_3, a_2, a_3) = 0$.

ES: $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in K^{2 \times 2}$

$$A = (a_{11} e_1 + a_{21} e_2, a_{12} e_1 + a_{22} e_2) \Rightarrow$$

$$\det A = \det(a_{11} e_1, a_{22} e_2) + \det(a_{21} e_2, a_{12} e_1) + \dots$$

$$= a_{11} a_{22} - a_{21} a_{12}$$

$A = \begin{pmatrix} a_{11} & & 0 \\ & \dots & \\ 0 & & a_{nn} \end{pmatrix} = \text{diag}(a_{11}, \dots, a_{nn}) \in K^{n \times n}$
 (MATRICE DIAGONALE)

$$\det A = \det(a_{11} e_1, \dots, a_{nn} e_n) = a_{11} \dots a_{nn}$$

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \in K^{3 \times 3}$ (MATRICE TRIANGOLARE SUPERIORE)

$$\begin{aligned}
\det A &= \det(a_{11}e_1, a_{12}e_1 + a_{22}e_2, a_{13}e_1 + a_{23}e_2 + a_{33}e_3) \\
&= \det(a_{11}e_1, a_{12}e_1, a_{33}e_3) + \det(a_{11}e_1, a_{22}e_2, a_{33}e_3) \\
&= \det(a_{11}e_1, a_{22}e_2, a_{13}e_1) + \det(a_{11}e_1, a_{22}e_2, a_{23}e_2) \\
&\quad + \det(a_{11}e_1, a_{22}e_2, a_{33}e_3) \\
&= a_{11}a_{22}a_{33}
\end{aligned}$$

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(Zusatz)
 $K \rightarrow K$
 $A = (a_{ij}) \in K^{n \times n}$
 $v \in K^n, v \neq 0, v^T A = 0 \iff \det A = 0$
 $v \in \ker A, v \neq 0 \iff \det A = 0$
 $(a_{11}, \dots, a_{1n}) \cdot v = 0, \dots, (a_{n1}, \dots, a_{nn}) \cdot v = 0$
 $\iff \sum_{j=1}^n a_{ij} v_j = 0, i=1, \dots, n$
 $\iff \sum_{j=1}^n v_j (a_{1j}, \dots, a_{nj}) = 0$
 $\iff \sum_{j=1}^n v_j \cdot \text{row } j = 0$
 $\iff \text{row } 1 \cdot v_1 + \dots + \text{row } n \cdot v_n = 0$
 $\iff \text{row } 1 \cdot v_1 = -(\text{row } 2 \cdot v_2 + \dots + \text{row } n \cdot v_n)$
 $\iff \text{row } 1 \cdot v_1 = 0$
 $\iff v_1 = 0$
 $\iff v = 0$

LEMMA 3.1
 Sei $A \in K^{n \times n}$. Dann gilt:
 (i) $\det A = 0 \iff \exists v \in K^n, v \neq 0, Av = 0$
 (ii) $\det A = 0 \iff \exists w \in K^n, w \neq 0, w^T A = 0$
 (iii) $\det A = 0 \iff \exists i, j \in \{1, \dots, n\}, a_{ij} = 0$
 (iv) $\det A = 0 \iff \exists i, j \in \{1, \dots, n\}, a_{ij} = 0$

$$\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1, \det \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0, \det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1, \det \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0$$

LEMMA 3.2
 Sei $A = (a_{ij}) \in K^{n \times n}$. Dann gilt:
 (i) $\det A = 0 \iff \exists i, j \in \{1, \dots, n\}, a_{ij} = 0$
 (ii) $\det A = 0 \iff \exists i, j \in \{1, \dots, n\}, a_{ij} = 0$
 (iii) $\det A = 0 \iff \exists i, j \in \{1, \dots, n\}, a_{ij} = 0$
 (iv) $\det A = 0 \iff \exists i, j \in \{1, \dots, n\}, a_{ij} = 0$