* DETERMINANTE *

def (determinante):

$\mathbb{K}$ campo,

$$F : \mathbb{K}^{n\times n} \to \mathbb{K}$$

t.c. $\forall A = (a_1, \dots, a_n) \in \mathbb{K}^{n\times n}$:

$\boxed{D1}$ $\forall t \in \mathbb{K}$, $\forall j \in \{1,\dots,n\}$: $F(a_1,\dots, t a_j, \dots, a_n) = t\, F(a_1,\dots, a_j, \dots, a_n)$

$\boxed{D2}$ $\forall j \in \{1,\dots,n\}$, $b \in \mathbb{K}^n$:

$$F(a_1,\dots, a_j + b, \dots, a_n) = F(a_1,\dots,a_j,\dots,a_n) + F(a_1,\dots,b,\dots,a_n)$$

$\boxed{D3}$ se $\exists\, i \neq j \in \{1,\dots,n\}$ t.c. $a_i = a_j$ allora $F(a_1,\dots,a_n) = 0$

$\boxed{D4}$ $F(e_1,\dots,e_n) = 1$

TEO: $\forall n$ intero ≥ 1, $\exists!\ F : \mathbb{K}^{n\times n} \to \mathbb{K}$ che verifica $\boxed{D1}, \dots, \boxed{D4}$.

Tale funz si dice DETERMINANTE e si indica con det.

Es: $\det\begin{pmatrix}1&0\\0&1\end{pmatrix} = 1$; $\det\begin{pmatrix}2&0\\0&1\end{pmatrix} = \det(2e_1, e_2) = 2\det\begin{pmatrix}1&0\\0&1\end{pmatrix} = 2$;

$$\det\begin{pmatrix}1&1\\0&1\end{pmatrix} = \det(e_1, e_1+e_2) = \underbrace{\det(e_1,e_2)}_{1} + \underbrace{\det(e_1,e_1)}_{0} = 1$$

Oss: Sia $A = (a_1,\dots,a_n) \in \mathbb{K}^{n\times n}$

1) SE $\exists j$ t.c. $a_j = 0$ allora $\det A = 0$ ($\boxed{D1}\dots$)

2) $\forall i \neq j$: $\det(a_1,\dots,a_i,\dots,a_j,\dots,a_n) = -\det(a_1,\dots,a_j,\dots,a_i,\dots,a_n)$
(scambio righe $\Rightarrow$ cambio segno)

Es: $\det(a_1+a_2, a_1+a_2) = 0$

$\| (\boxed{D2}\dots)$

$$\underbrace{\det(a_1,a_1)}_{=0\ \boxed{D3}} + \det(a_2,a_1) + \det(a_1,a_2) + \underbrace{\det(a_2,a_2)}_{=0\ \boxed{D3}}$$

$$\Rightarrow \det(a_2,a_1) = -\det(a_1,a_2)$$

(3) SE $a_1,\dots,a_n$ sono lin. dip. allora $\det A = 0$.

Es: $a_1 = a_2 + 3a_3 \Rightarrow \det(a_1,a_2,a_3) = \det(a_2+3a_3, a_2, a_3)$
$= \det(a_2,a_2,a_3) + 3\det(a_3,a_2,a_3) = 0$.

Es: • $A = \begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix} \in \mathbb{K}^{2\times 2}$

$$A = (a_{11}e_1 + a_{21}e_2,\ a_{12}e_1 + a_{22}e_2) \Rightarrow$$

$$\det A = \det(a_{11}e_1, a_{22}e_2) + \det(a_{21}e_2, a_{12}e_1) + \dots$$
$$= a_{11}a_{22} - a_{21}a_{12}$$

• $A = \begin{pmatrix}a_{11}& & 0\\ &\ddots& \\ 0& & a_{nn}\end{pmatrix} = \mathrm{diag}(a_{11},\dots,a_{nn}) \in \mathbb{K}^{n\times n}$

(MATRICE DIAGONALE)

$$\det A = \det(a_{11}e_1,\dots,a_{nn}e_n) = a_{11}\cdots a_{nn}$$

• $A = \begin{bmatrix}a_{11}&a_{12}&a_{13}\\0&a_{22}&a_{23}\\0&0&a_{33}\end{bmatrix} \in \mathbb{K}^{3\times 3}$ (MATRICE TRIANGOLARE SUPERIORE

$$\det A = \det(a_{11}e_1,\ a_{12}e_1 + a_{22}e_2,\ a_{13}e_1 + a_{23}e_2 + a_{33}e_3)$$

$$= \underbrace{\det(a_{11}e_1,\ a_{12}e_1,\ a_3)}_{=0} + \det(a_{11}e_1,\ a_{22}e_2,\ a_3)$$

$$= \underbrace{\det(a_{11}e_1,\ a_{22}e_2,\ a_{13}e_1)}_{=0} + \underbrace{\det(a_{11}e_1,\ a_{22}e_2,\ a_{23}e_2)}_{=0}$$

$$+ \det(a_{11}e_1,\ a_{22}e_2,\ a_{33}e_3)$$

$$= a_{11}a_{22}a_{33}.$$