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* ALGEBRE di MATRICI *

• PRODOTTO tra matrici

$\mathbb{K}$ campo ; $A \in \mathbb{K}^{n \times m}$, $B \in \mathbb{K}^{p \times q}$

<u>def</u>: SE $m = p$, si def $AB \in \mathbb{K}^{n \times q}$ come dall'es seguente:

$A \in \mathbb{K}^{3 \times 2}$, $B \in \mathbb{K}^{2 \times 4}$ ...

$$AB = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & \cdots & a_{11}b_{14} + a_{12}b_{24} \\ a_{21}b_{11} + a_{22}b_{21} & \cdots & a_{21}b_{14} + a_{22}b_{24} \\ a_{31}b_{11} + a_{32}b_{21} & \cdots & a_{31}b_{14} + a_{32}b_{24} \end{pmatrix} \in \mathbb{K}^{3 \times 4}$$

SE $m \neq p$ non si def il prodotto di A per B.

<u>Es</u>: • $a \in \mathbb{K}^{1 \times m}$, $b \in \mathbb{K}^{m \times 1} = \mathbb{K}^m$

$(a_1, \dots, a_m)$, $\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$, $a_k, b_k \in \mathbb{K}$

$$(a_1, \dots, a_m) \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = a_1 b_1 + a_2 b_2 + \dots + a_m b_m \in \mathbb{K}^{1 \times 1} = \mathbb{K}$$

• $A \in \mathbb{K}^{n \times m}$, $b \in \mathbb{K}^{m \times 1} = \mathbb{K}^m$

$(a_1, \dots, a_m)$, $\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$, $a_k \in \mathbb{K}^n$, $b_k \in \mathbb{K}$

$$Ab = b_1 a_1 + \dots + b_m a_m \in \mathbb{K}^{n \times 1} = \mathbb{K}^n$$

<u>Oss</u>: Ab è la comb lin delle <u>colonne</u> di A a coeff le <u>componenti</u> di b.

• $a \in \mathbb{K}^{1 \times m}$, $B \in \mathbb{K}^{m \times n}$

$(a_1, \dots, a_m)$, $\begin{pmatrix} b'_1 \\ \vdots \\ b'_m \end{pmatrix}$, $a_k \in \mathbb{K}$, $b'_k \in \mathbb{K}^{1 \times n}$

$$aB = a_1 b'_1 + \dots + a_m b'_m \in \mathbb{K}^{1 \times n}$$

<u>Oss</u>: aB è la comb lin delle <u>righe</u> di B a coeff le <u>componenti</u> di a.

<u>Es</u>: $A \in \mathbb{K}^{3 \times 2}$, $B \in \mathbb{K}^{2 \times 4}$

• $A = (a_1, a_2) = \begin{pmatrix} a'_1 \\ a'_2 \\ a'_3 \end{pmatrix}$, $B = (b_1, b_2, b_3, b_4) = \begin{pmatrix} b'_1 \\ b'_2 \end{pmatrix}$

Verificare che:

$$AB = \begin{pmatrix} a'_1 b_1 & a'_1 b_2 & a'_1 b_3 & a'_1 b_4 \\ a'_2 b_1 & a'_2 b_2 & a'_2 b_3 & a'_2 b_4 \\ a'_3 b_1 & a'_3 b_2 & a'_3 b_3 & a'_3 b_3 \end{pmatrix}$$ "PRODOTTO RIGHE per COLONNE"

$$= (Ab_1, Ab_2, Ab_3, Ab_4)$$ "prodotto calcolato per colonne"

$$= \begin{pmatrix} a'_1 B \\ a'_2 B \\ a'_3 B \end{pmatrix}$$ "prodotto calcolato per righe"

Es: • $A = (a_1, a_2, a_3, a_4) \in \mathbb{C}^{3\times 4}$, $a_k \in \mathbb{C}^3$

$$A\begin{pmatrix} i \\ 3 \\ 0 \\ 1+i \end{pmatrix} = i a_1 + 3a_2 + (1+i) a_4 \in \mathbb{C}^3$$

• $B = \begin{pmatrix} b'_1 \\ b'_2 \end{pmatrix} \in \mathbb{C}^{2\times 4}$, $b'_k \in \mathbb{C}^{1\times 4}$

$$\begin{pmatrix} 1 & i \\ 0 & 1+i \\ i & 6 \end{pmatrix} B = \begin{pmatrix} b'_1 + i b'_2 \\ (1+i)\, b'_2 \\ i b'_1 + 6 b'_2 \end{pmatrix} \in \mathbb{C}^{3\times 4}$$

• determinare M, N t.c.

1) $AM = (3a_1,\ a_2 - 3a_3,\ i a_3 - a_4)$

2) $NB = \begin{pmatrix} b'_1 + b'_2 \\ b'_1 \\ (3+i)\, b'_2 \end{pmatrix}$

• PROPRIETÀ del prodotto tra matrici:

associativa: $A \in \mathbb{K}^{m\times n}$, $B \in \mathbb{K}^{n\times q}$, $C \in \mathbb{K}^{q\times p}$

allora ...

$$\underset{m\times n}{A}\,(\underset{n\times p}{BC}) = (\underset{m\times q}{AB})\,\underset{q\times p}{C} \in \mathbb{K}^{m\times p}$$

elementi neutri: r intero > 0,

$$I_r = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = (\underbrace{e_1, e_2, \dots, e_r}_{\text{base canonica di } \mathbb{K}^r}) \in \mathbb{K}^{r\times r}$$

allora: $\forall A \in \mathbb{K}^{n\times m}$ si ha:

$$A I_m = A, \quad I_n A = A$$

I_r si dice <u>la matrice identica di ordine</u> r

linearità a destra: $A \in \mathbb{K}^{n\times m}$, $B, C \in \mathbb{K}^{m\times q}$, $a \in \mathbb{K}$

si ha:

$$A(B+C) = AB + AC$$
$$A(aB) = a(AB)$$

Es: $A \in \mathbb{C}^{4\times 3}$, $b, c \in \mathbb{C}^3$

$$\begin{aligned} A(b+c) &= (a_1, a_2, a_3)(b+c) \\ &= (b_1+c_1)\,a_1 + (b_2+c_2)\,a_2 + (b_3+c_3)\,a_3 \\ &= b_1 a_1 + c_1 a_1 + \dots + b_3 a_3 + c_3 a_3 \\ &= (b_1 a_1 + b_2 a_2 + b_3 a_3) + (c_1 a_1 + c_2 a_2 + c_3 a_3) \\ &= Ab + Ac \end{aligned}$$

linearità a sinistra: $A, B \in \mathbb{K}^{n\times m}$, $C \in \mathbb{K}^{m\times q}$, $a \in \mathbb{K}$

si ha:

$$(A+B)\,C = AC + BC$$
$$(aA)\,C = a(AC)$$

non commutatività:

- $A \in \mathbb{K}^{3\times 5}$, $B \in \mathbb{K}^{5\times 7}$

  $\Rightarrow$ AB è def, BA no

- $A \in \mathbb{K}^{3\times 5}$, $B \in \mathbb{K}^{5\times 3}$

  $\Rightarrow$ $AB \in \mathbb{K}^{3\times 3}$, $BA \in \mathbb{K}^{5\times 5}$

  e $AB \neq BA$

- $A \in \mathbb{K}^{3\times 3}$, $B \in \mathbb{K}^{3\times 3}$

  $\Rightarrow$ $AB, BA \in \mathbb{K}^{3\times 3}$

Es
- $A = I_3 \Rightarrow I_3 B = B = B I_3$

  (cioè $AB = BA$)

- $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

  $AB = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $BA = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

  (cioè $AB \neq BA$)

- TRASPOSIZIONE: operatori T e H

$$A = \begin{pmatrix} a_{11} & \cdots & a_{15} \\ \vdots & & \vdots \\ a_{31} & \cdots & a_{35} \end{pmatrix} \in \mathbb{K}^{3\times 5}$$

def: $$A^T = \begin{pmatrix} a_{11} & \cdots & a_{31} \\ \vdots & & \vdots \\ a_{15} & \cdots & a_{35} \end{pmatrix} \in \mathbb{K}^{5\times 3}$$

A^T si dice la trasposta di A.

Es: $A = \begin{pmatrix} 3 & 2 \\ 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathbb{Q}^{3\times 2}$

$\Rightarrow A^T = \begin{pmatrix} 3 & 0 & 1 \\ 2 & -1 & 0 \end{pmatrix} \in \mathbb{Q}^{2\times 3}$

Proprietà di T:

- $\forall\, A, B \in \mathbb{K}^{n\times m}$, $a \in \mathbb{K}$...

$$\boxed{\begin{aligned} (A+B)^T &= A^T + B^T \\ (aA)^T &= a(A^T) \\ (A^T)^T &= A \end{aligned}}$$

- $\forall\, A \in \mathbb{K}^{n\times m}$, $B \in \mathbb{K}^{m\times q}$

  $AB \in \mathbb{K}^{n\times q}$, $(AB)^T \in \mathbb{K}^{q\times n}$

  $A^T \in \mathbb{K}^{m\times n}$, $B^T \in \mathbb{K}^{q\times m}$, $B^T A^T \in \mathbb{K}^{q\times n}$

  e $\boxed{(AB)^T = B^T A^T}$

Es: $A = \begin{pmatrix} 3 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathbb{Q}^{3\times 2}$, $B = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \in \mathbb{Q}^{2\times 2}$

calcolare AB, $(AB)^T$, $B^T A^T$.

$$A = \begin{pmatrix} a_{11} & \cdots & a_{15} \\ a_{21} & \cdots & a_{25} \end{pmatrix} \in \mathbb{C}^{2\times 5}$$

def: $\overline{A} = \begin{pmatrix} \overline{a_{11}} & \cdots & \overline{a_{15}} \\ \overline{a_{21}} & \cdots & \overline{a_{25}} \end{pmatrix} \in \mathbb{C}^{2\times 5}$

$$A^H = (\overline{A})^T = \overline{(A^T)} \in \mathbb{C}^{5\times 2}$$

$\overline{A}$ si dice la coniugata di A

A^H si dice la trasposta coniugata di A

Proprieta':

- $\overline{(A+B)} = \overline{A} + \overline{B}$, $\overline{(cA)} = \overline{c}\,\overline{A}$, $\overline{(\overline{A})} = A$

  $\overline{(AB)} = \overline{A}\,\overline{B}$;

- $(A+B)^H = A^H + B^H$, $(cA)^H = \overline{c} A^H$, $(A^H)^H = A$

  $(AB)^H = B^H A^H$

Oss: $a, b \in \mathbb{R}^n$ con ps canonico $\Rightarrow a \cdot b = b^T a$

$a, b \in \mathbb{C}^n$ con ph canonico $\Rightarrow a \cdot b = b^H a$