Oss. $A \in \mathbb{C}^{n \times n}$, $\sigma_1, \dots, \sigma_n$ v.s. di A

U, Σ, V decomp ai v.s. di A

- $U, \Sigma, V \in \mathbb{C}^{n \times n}$ t.c. $A = U \Sigma V^H$

$$\Rightarrow |\det A| = |\det U \det \Sigma \det(V^H)| = \sigma_1 \cdots \sigma_n$$

$$\boxed{A \text{ invert} \Leftrightarrow \text{tutti i vs sono } \neq 0}$$

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* STUDIO di SISTEMI di EQ LIN *

- $A \in \mathbb{C}^{n \times k}$; $U = (u_1, \dots, u_n)$, Σ, $V = (v_1, \dots, v_k)$ decomp vs

1) $\operatorname{rk} A = r$ (# vs non nulli)

2) $v_{r+1}, \dots, v_k$ base o n di $\ker A \subset \mathbb{C}^k$

3) $u_1, \dots, u_r$ base o n di $\operatorname{Im} A \subset \mathbb{C}^n$

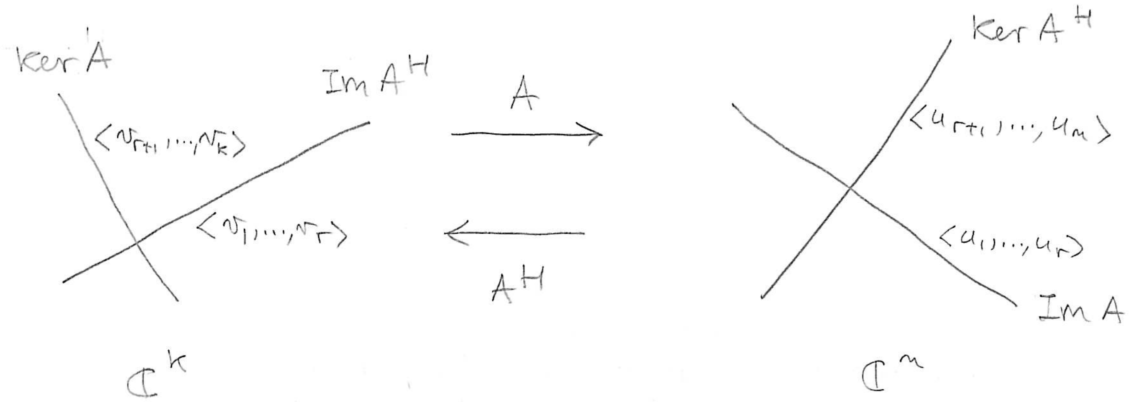
- V, Σ^H, U è decomp vs di A^H ($\Rightarrow$ vs A^H = vs A)

1) $\operatorname{rk} A^H = r$ (# vs non nulli)

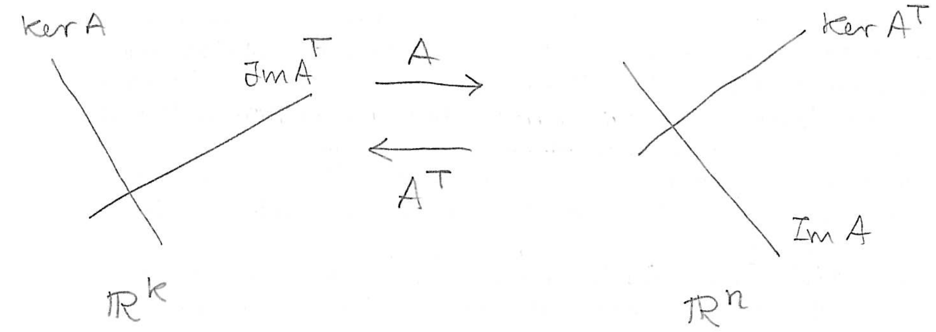
2) $u_{r+1}, \dots, u_n$ base o n di $\ker A^H \subset \mathbb{C}^n$

3) $v_1, \dots, v_r$ base o n di $\operatorname{Im} A^H \subset \mathbb{C}^k$

- $\mathbb{C}^k = \langle v_1, \dots, v_r \rangle \overset{\perp}{\oplus} \langle v_{r+1}, \dots, v_k \rangle = \operatorname{Im} A^H \overset{\perp}{\oplus} \ker A$
- $\mathbb{C}^n = \langle u_1, \dots, u_r \rangle \overset{\perp}{\oplus} \langle u_{r+1}, \dots, u_n \rangle = \operatorname{Im} A \overset{\perp}{\oplus} \ker A^H$



Caso "reale": $A \in \mathbb{R}^{n \times k}$; U, Σ, V ad elem reali...



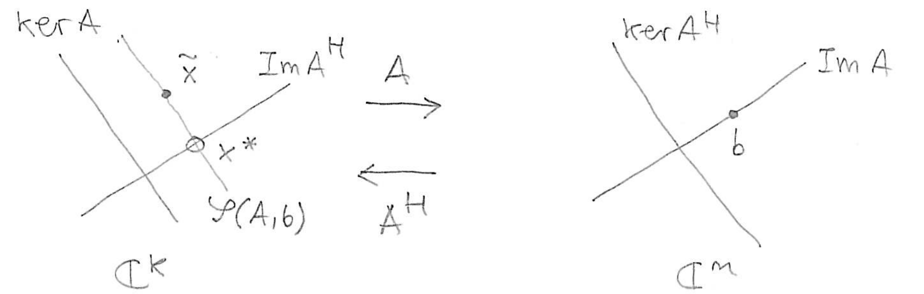
- $\mathbb{R}^k = \operatorname{Im} A^T \overset{\perp}{\oplus} \ker A$
- $\mathbb{R}^n = \operatorname{Im} A \overset{\perp}{\oplus} \ker A^T$

STRUTTURA dell'INS delle SOLUZ

- $A \in \mathbb{C}^{n \times k}$, $b \in \mathbb{C}^n$

(1) se $b \in \operatorname{Im} A$:

- $\mathcal{S}(A, b) \neq \emptyset$
- $\tilde{x} \in \mathcal{S}(A, b) \Rightarrow \mathcal{S}(A, b) = \tilde{x} + \ker A$



($\ker A \neq \{0\}$)

- $\mathcal{S}(A,b) \cap \operatorname{Im} A^H$ ha un solo elem (x^*)
- x^* è l'elem di $\mathcal{S}(A,b)$ di <u>minima norma euclidea</u>

(2) Se $b \notin \operatorname{Im} A \Rightarrow \mathcal{S}(A,b) = \emptyset$

<u>Es</u>: $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (in $\mathbb{R}$)

- $b \in \operatorname{Im} A$, $\tilde{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in \mathcal{S}(A,b)$
- $\mathcal{S}(A,b) = \tilde{x} + \operatorname{Ker} A = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \left\langle \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\rangle$

$$= \left\{ x = \begin{bmatrix} 1+\lambda \\ -\lambda \\ \lambda \end{bmatrix},\ \lambda \in \mathbb{R} \right\}$$

- $y \in \mathcal{S}(A,b)$, $\|y\|^2 = (1+\lambda)^2 + 2\lambda^2$

$$F(\lambda) = (1+\lambda)^2 + 2\lambda^2$$

F assume val minimo per $\lambda^* = -\frac{1}{3}$

e l'elem corrisp è $x^* = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

- $y \in \mathcal{S}(A,b) \cap \operatorname{Im} A^H \Leftrightarrow y \in \mathcal{S}(A,b)$ e $y \perp \operatorname{Ker} A$

$$\Leftrightarrow \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right) \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 0 \Leftrightarrow \lambda = -\frac{1}{3} \ \ldots$$

SOLUZIONI nel SENSO dei MINIMI QUADRATI

- $A \in \mathbb{C}^{m \times k}$, $b \in \mathbb{C}^m$

$F: \mathbb{C}^k \to \mathbb{R}$ t.c. $F(x) = \|Ax - b\|^2$

> <u>def</u> $y \in \mathbb{C}^k$ che rende minimo il valore di F, ovvero t.c. $\forall w \in \mathbb{C}^k$, $\|Ay - b\|^2 \leq \|Aw - b\|^2$ si chiama <u>SOLUZIONE di $Ax = b$ NEL SENSO dei MINIMI QUADRATI</u>
>
> L'ins delle soluz di $Ax=b$ nel senso dei m.q. si indica con $\mathcal{S}_{MQ}(A,b) \subset \mathbb{C}^k$

<u>Es</u>: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ (in $\mathbb{R}$)

- $b \notin \operatorname{Im} A$, $F: \mathbb{R}^2 \to \mathbb{R}$
- $F(x) = \|Ax - b\|^2 = \left\| \begin{bmatrix} x_1 \\ x_2 \\ -1 \end{bmatrix} \right\|^2 = x_1^2 + x_2^2 + (-1)^2$

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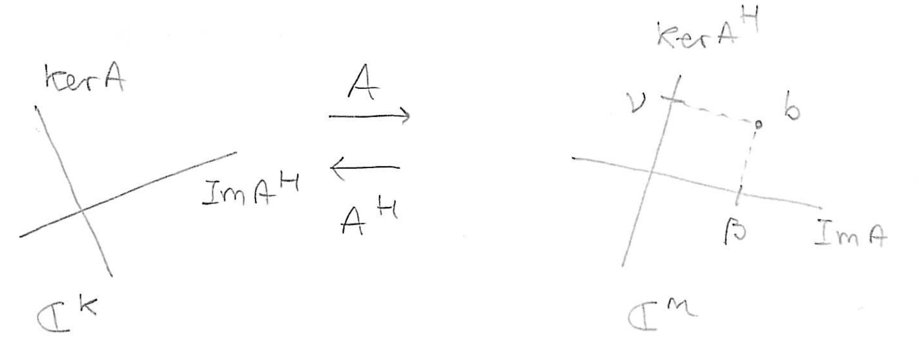
(1) <u>Se</u> $b \in \operatorname{Im} A$ <u>allora</u> $\mathcal{S}_{MQ}(A,b) = \mathcal{S}(A,b)$

(infatti: $x \in \mathcal{S}(A,b) \Rightarrow F(x) = 0$ e $F \geq 0 \ldots$)

(2) <u>se</u> $b \notin \operatorname{Im} A$:

- $\mathcal{S}(A,b) = \emptyset$
- $b = \beta + \nu$, $\beta \in \operatorname{Im} A$, $\nu \in \ker A^H$



Es: $A = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\end{bmatrix} \in \mathbb{R}^{3\times 4}$, $b = \begin{bmatrix}0\\1\\1\end{bmatrix} \in \mathbb{R}^3$

- $\operatorname{Im} A = \left\langle \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\rangle$, $\ker A = \left\langle \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\rangle$

- $\ker A^H = \left\langle \begin{bmatrix}0\\0\\1\end{bmatrix} \right\rangle$

- $b = \underbrace{\beta_1 \begin{bmatrix}1\\0\\0\end{bmatrix} + \beta_2 \begin{bmatrix}0\\1\\0\end{bmatrix}}_{\beta \in \operatorname{Im} A} + \underbrace{\nu_1 \begin{bmatrix}0\\0\\1\end{bmatrix}}_{\nu \in \ker A^H}$

$\Rightarrow \beta_1 = 0, \ \beta_2 = \nu_1 = 1$ ovvero

$b = \begin{bmatrix}0\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix} = \beta + \nu$

- $\mathcal{S}_{MQ}(A,b) = \mathcal{S}(A,\beta)$

$\forall x \in \mathbb{C}^k$: $\|Ax-b\|^2 = \|Ax-\beta-\nu\|^2$

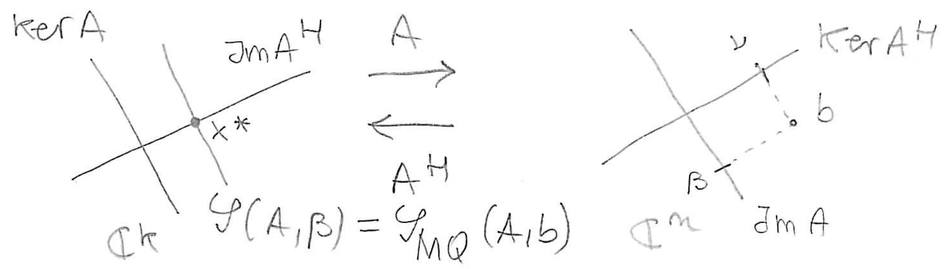
$= \|Ax-\beta\|^2 + \|\nu\|^2$

dunque $\|Ax-b\|^2$ assume valore minimo

$\Leftrightarrow$ $\|Ax-\beta\|^2$ assume valore minimo

ovvero $\mathcal{S}_{MQ}(A,b) = \mathcal{S}_{MQ}(A,\beta) = \mathcal{S}(A,\beta)$

$\boxed{\text{perché } \beta \in \operatorname{Im} A}$



- $\mathcal{S}(A,\beta) \cap \operatorname{Im} A^H$ ha un solo elemento, x^*, ed è l'elem di $\mathcal{S}_{MQ}(A,b)$ di minima norma euclidea.

Es: $A = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\end{bmatrix} \in \mathbb{R}^{3\times 4}$, $b = \begin{bmatrix}0\\1\\1\end{bmatrix} \in \mathbb{R}^3$

determ $\mathcal{S}_{MQ}(A,b)$.

- $b \notin \operatorname{Im} A \Rightarrow \mathcal{S}(A,b) = \emptyset$
- $b = \begin{bmatrix}0\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix} = \beta + \nu$
- $\tilde{x} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \in \mathcal{S}(A,\beta)$

$\Rightarrow \mathcal{S}_{MQ}(A,b) = \mathcal{S}(A,\beta) = \overset{\tilde{x}}{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}} + \overset{\ker A}{\left\langle \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\rangle}$

- $x \in \mathcal{S}_{MQ}(A,b) \cap \operatorname{Im} A^H \Leftrightarrow$

$x \in \mathcal{S}_{MQ}(A,b)$ e $x \perp \ker A$

$\Leftrightarrow x = \begin{bmatrix}0\\1\\0\\0\end{bmatrix} + \alpha_1 \begin{bmatrix}0\\0\\1\\0\end{bmatrix} + \alpha_2 \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$ e $x \cdot \begin{bmatrix}0\\0\\1\\0\end{bmatrix} = x \cdot \begin{bmatrix}0\\0\\0\\1\end{bmatrix} = 0$

$\Leftrightarrow \alpha_1 = 0, \ \alpha_2 = 0$ quindi $\boxed{x^* = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}}$