Es: $B = \begin{bmatrix}1&0&1\\1&0&-1\end{bmatrix} \in \mathbb{C}^{2\times 3}$; $U \in \mathbb{C}^{2\times 2}$, $V \in \mathbb{C}^{3\times 3}$

- $B^H B = \begin{bmatrix}2&0&0\\0&0&0\\0&0&2\end{bmatrix}$; $\lambda_1 = \lambda_2 = 2$, $\lambda_3 = 0$
- $\ker(B^H B - 2I) = \ker \begin{bmatrix}0&0&0\\0&-2&0\\0&0&0\end{bmatrix} = \left\langle \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\rangle$

  $\ker(B^H B) = \ker \begin{bmatrix}2&0&0\\0&0&0\\0&0&2\end{bmatrix} = \left\langle \begin{bmatrix}0\\1\\0\end{bmatrix} \right\rangle$
- $B^H B = \underbrace{\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}}_{V} \begin{bmatrix}2&0&0\\0&2&0\\0&0&0\end{bmatrix} V^{-1}$ | altra scelta di V: $\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$
- $p = 2$, $\sigma_1 = \sigma_2 = \sqrt{2}$, $\Sigma = \begin{bmatrix}\sqrt{2}&0&0\\0&\sqrt{2}&0\end{bmatrix}$
- $u_1 = \frac{1}{\sigma_1} B v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix}1&0&1\\1&0&-1\end{bmatrix} \begin{bmatrix}1\\0\\0\end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix}1\\1\end{bmatrix}$

  $u_2 = \frac{1}{\sigma_2} B v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix}1&0&1\\1&0&-1\end{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix}1\\-1\end{bmatrix}$

  $\Rightarrow U = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}$

Es: $A = \begin{bmatrix}0&1\\0&0\end{bmatrix} \in \mathbb{C}^{2\times 2}$; determ d. v.s di A.

Oss: $A \in \mathbb{C}^{n\times k}$; $U = (u_1, \dots, u_n)$, Σ, $V = (v_1, \dots, v_k)$ decompos ai v.s. di A ; $\sigma_1, \dots, \sigma_r$ v.s non nulli di A.

- $\Sigma = \begin{bmatrix}\sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & \end{bmatrix}$; $\operatorname{Ker} \Sigma = \langle e_{r+1}, \dots, e_k \rangle \subset \mathbb{C}^k$

  $\operatorname{Im} \Sigma = \langle e_1, \dots, e_r \rangle \subset \mathbb{C}^n$
- $x \in \operatorname{Ker} A \Leftrightarrow x' \in \operatorname{Ker} \Sigma$

  $e_{r+1} \in \operatorname{Ker} \Sigma \Rightarrow v_{r+1} \in \operatorname{Ker} A$

  $\vdots$

  $e_k \in \operatorname{Ker} \Sigma \Rightarrow v_k \in \operatorname{Ker} A$

  e $\operatorname{Ker} A = \langle v_{r+1}, \dots, v_k \rangle$ (infatti ...)

  e $v_{r+1}, \dots, v_k$ o.n $\Rightarrow$ base on di $\operatorname{Ker} A$
- $y \in \operatorname{Im} A \Leftrightarrow y' \in \operatorname{Im} \Sigma$

  $e_1 \in \operatorname{Im} \Sigma \Rightarrow u_1 \in \operatorname{Im} A$

  $\vdots$

  $e_r \in \operatorname{Im} \Sigma \Rightarrow u_r \in \operatorname{Im} A$

  e $\operatorname{Im} A = \langle u_1, \dots, u_r \rangle$

  e $u_1, \dots, u_r$ o.n $\Rightarrow$ base on di $\operatorname{Im} A$
- $\dim \operatorname{Ker} A = \dim \operatorname{Ker} \Sigma = k - r$

  $\underbrace{\dim \operatorname{Im} A}_{\operatorname{rk} A} = \underbrace{\dim \operatorname{Im} \Sigma}_{\operatorname{rk} \Sigma} = r$

Es: $A = (u_1, u_2) \begin{bmatrix}2&0&0\\0&0&0\end{bmatrix} (v_1, v_2, v_3)^H$

- $\operatorname{rk} A = \dim \operatorname{Im} A = 1$; $\dim \operatorname{Ker} A = 2$
- $\operatorname{Im} A = \langle u_1 \rangle$, $\operatorname{Ker} A = \langle v_2, v_3 \rangle$