Note Title 12/10/2021

$$\int \frac{x}{\sqrt{1+x^2}} \lg x \, dx = \int \frac{2x}{2\sqrt{1+x^2}} \lg x =$$

$$= (\lg x)\sqrt{1+x^2} - \int \frac{1}{x}\sqrt{1+x^2}$$

$x = \sinh t$

$dx = \cosh t \, dt$

$$\int \frac{\sqrt{1+x^2}}{x} dx = \int \frac{\cosh t}{\sinh t} \cosh t \, dt =$$

$$= \int \frac{\frac{e^{2t}+1+e^{-2t}}{4}2}{\frac{e^t - e^{-t}}{2}} dt = \frac{1}{2}\int \frac{e^{2t}+1+e^{-2t}}{e^t - e^{-t}} dt$$

$e^t = u$

$t = \lg u$

$dt = \frac{1}{u} du$

$2x = (1+x^2)'$

$$\int \frac{2x \, dx}{2\sqrt{1+x^2}} \quad \boxed{y = 1+x^2} \quad dy$$

$$\int \frac{1}{2\sqrt{y}} dy =$$

$$\sqrt{y}\Big|_{y=1+x^2} =$$

$$= \sqrt{1+x^2}$$

$$= \frac{1}{2} \int \frac{u^2 + 1 + \frac{1}{u^2}}{u - \frac{1}{u}} \frac{1}{u} du =$$

$$= \frac{1}{2} \int \frac{\frac{u^4 + u^2 + 1}{u^2}}{\frac{u^2 - 1}{u}} \cdot \frac{1}{u} du = \frac{1}{2} \int \frac{u^4 + u^2 + 1}{u^4 - u^2} du$$

Si divide e il denom. ha zeri $u = 0$ doppi e 1 e -1 semplici.

$$\int \sqrt{\frac{1-x}{1+x}}\,dx = \left[\sqrt{\frac{1-x}{1+x}} = t\right]$$

$$\frac{1-x}{1+x} = t^2$$

$$1-x = t^2+xt^2$$

$$1-t^2 = x(1+t^2)$$

$$x = \frac{1-t^2}{1+t^2}$$

$$dx = \frac{-2t(1+t^2)-2t(1-t^2)}{(1+t^2)^2}\,dt$$

$$\frac{-2t-2t^3-2t+2t^3}{(1+t^2)^2}$$

$$= \int t\,\frac{-4t}{(1+t^2)^2}\,dt =$$

$$= -2\int t\,\frac{2t}{(1+t^2)^2}\,dt$$

$$\int f\left(x, \sqrt{\frac{ax+b}{cx+d}}\right)dx$$

$$\int \frac{\sqrt{x}}{\sqrt{x}+1}\,dx = \left|\begin{array}{l}\sqrt{x}=t\\ x=t^2\\ dx=2t\,dt\end{array}\right| = \int \frac{t}{t+1}\,2t\,dt = 2\int \frac{t^2}{t+1}\,dt = 2\left[\frac{t^2}{2}-t+\ln|t+1|\right]$$

$$\begin{array}{r|l} t^2 & t+1 \\ \hline t^2+t & t-1 \\ \hline =-t & \\ -t-1 & \\ \hline 1 & \end{array}$$

$$\frac{t^2}{t+1} = (t-1)+\frac{1}{t+1}$$

$$\int \frac{1}{x\sqrt{1-x^2}}\,dx = \left|\begin{array}{l}x=\sin t\\ t=\arcsin x\\ dx=\cos t\,dt\end{array}\right| = \int \frac{\cos t\,dt}{\sin t\,|\cos t|} =$$

$$= \operatorname{sgn}(\cos t)\int \frac{1}{\sin t}\,dt = \left|u=\operatorname{tg}\frac{t}{2}\right| = \int \frac{1+u^2}{2u}\cdot\frac{2}{1+u^2} = \ln|u|\Big|_{u=\operatorname{tg}\frac{t}{2}} = \ln\left|\operatorname{tg}\frac{t}{2}\right|\Big|_{t=\arcsin x} =$$

$$= \ln\left|\operatorname{tg}\frac{\arcsin x}{2}\right|$$

$$-\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \int\left[\frac{1-x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}\right]dx =$$

$$= \int \sqrt{1-x^2}\,dx - \int \frac{1}{\sqrt{1-x^2}}\,dx$$

$\arcsin x$ jedné $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$

$x = \sin t$

$\int |\cos t| \cos t\,dt$ zjednodušíš $\cos^2 t = \frac{1}{2} + \frac{1}{2}\cos 2t$

$\cos 2t = 2\cos^2 t - 1$

$$\int \frac{\sqrt{x}}{(1+x)^2}\,dx \;\overset{\sqrt{x}=t}{\underset{x=t^2,\ dx=2t\,dt}{=}}\; \int \frac{t}{(1+t^2)^2}\,2t\,dt = \int \frac{2t}{(1+t^2)^2}\,t\,dt =$$

$f(x, \sqrt{x})$

$$= t\left(-\frac{1}{1+t^2}\right) - \int 1\cdot\left(-\frac{1}{1+t^2}\right)dt =$$

$$= -\frac{t}{1+t^2} + \operatorname{arctg} t$$

$$\int \frac{2t}{(1+t^2)^2}\,dt \;\overset{1+t^2=y}{=}\; \int \frac{dy}{y^2} = -\frac{1}{y}\Big|_{y=1+t^2} =$$

$$dy = 2t\,dt$$

$$= -\frac{1}{1+t^2}$$

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx =$$

$$= e^x \sin x - \left[e^x (-\cos x) - \int e^x (-\cos x) \right] =$$

$$\int e^x \cos x \, dx = e^x (\sin x + \cos x) - \int e^x \cos x$$

$$2 \int e^x \cos x = e^x (\sin x + \cos x)$$

$$\int \frac{e^x}{(1+e^x)^2}\,dx = \begin{matrix} e^x = t \\ x = \ln t \\ dx = \frac{1}{t}\,dt \end{matrix} \int \frac{\cancel{t}}{(1+t)^2}\,\frac{1}{\cancel{t}}\,dt = \left| \frac{e^x}{(1+e^x)^2} = \frac{1}{(1+t)^2} \right|_{t=e^x}$$

$$= \int \frac{1}{(1+t)^2}\,dt = -\frac{1}{(1+t)}\Big|_{t=e^x} = -\frac{1}{1+e^x}$$

TUTTE le primitive sono $-\dfrac{1}{1+e^x} + C$

$e^x > 0$

$$\int_0^{2\pi} \sin kx \cos hx \, dx = 0 \qquad \forall\, h \neq k$$

$$\sin kx \cos hx = \frac{1}{2}\Big[\sin(h+k)x$$

$$\sin(hx+kx) = \sin hx\cos kx + \cos hx \sin kx$$

$$\sin(hx-kx) = \sin hx \cos kx - \cos hx \sin kx$$

$$= 2 \sin kx \cos hx =$$

$$= \sin(h+k)x - \sin(h-k)x$$

$$= \frac{1}{2}\int_0^{2\pi} \Big[\sin(h+k)x - \sin(h-k)x\Big] dx =$$

$$= \frac{1}{2}\left[\frac{-\cos(h+k)x}{h+k} - \frac{-\cos(h-k)x}{h-k}\right]_0^{2\pi}$$

$$= \underset{h\neq k}{-\frac{1}{h+k}} \cos(h+k)x \Big|_0^{2\pi} = 1-1=0$$

costo 0

$$\int_0^{2\pi} \sin^2 hx\,dx = \qquad h \in \mathbb{N}$$

$$= \int_0^{2\pi} \frac{1}{2}\,dx - \int_0^{2\pi} \frac{1}{2}\cos 2hx\,dx = 0$$

$$\frac{\sin 2hx}{2h}\Big|_0^{2\pi} = 0$$

$$= \pi$$

$$\sin^2 x = \frac{1}{2} - \frac{\cos 2x}{2}$$

$$\|\sin hx\| = \sqrt{\pi}$$

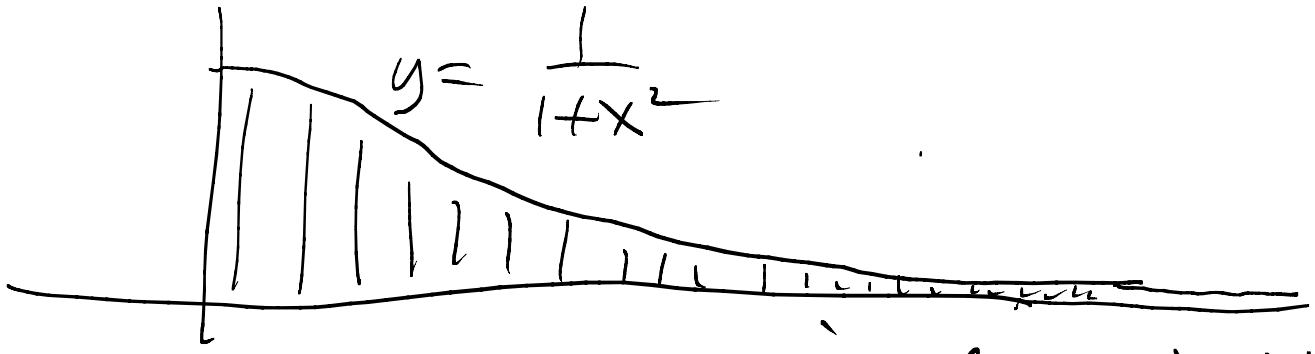
"$L^2(0,2\pi)$"

$$|a| = \sqrt{a\bar{a}}$$

INTEGRALI "IMPROPRI"

$\sum (t_{i+1} - t_i) \inf_{[t_i, t_{i+1}]} f$ $\qquad$ $\sum (t_{i+1} - t_i) \sup_{[t_i, t_{i+1}]} f$

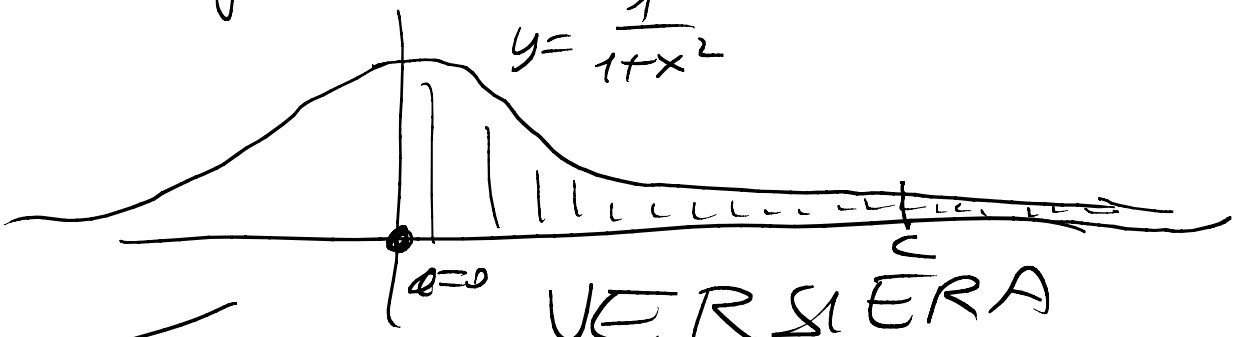
(1) COSA FARE se f NON è limitata?

(2) 

COSA FARE SE il dominio NON è limitato?

$$\int_a^{+\infty} f(t)\,dt \equiv \lim_{c \to +\infty} \int_a^c f(t)\,dt$$

f sia integrabile su ogni intervallo $[a, c]$, $c > a$.



VERSIERA di Maria Gaetana Agnesi

f è continua su $\mathbb{R}$ e quindi integrabile su ogni intervallo chiuso e limitato

$$\int_{-\infty}^{b} f(t)\,dt = \lim_{c \to -\infty} \int_{c}^{b} f(t)\,dt$$

$$\int_{-\infty}^{+\infty} f(t)\,dt \equiv \int_{-\infty}^{+\infty} f + \int_{-\infty}^{a} f$$

a ad arbitrio

$c \to -\infty$, $a \to +\infty$

$\lim_{\substack{b \to +\infty \\ a \to -\infty}} \int_{a}^{b} f$?

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx = \lim_{a \to +\infty} \int_{0}^{a} \frac{1}{1+x^2}\,dx +$$

$$+ \lim_{b \to -\infty} \int_{b}^{0} \frac{1}{1+x^2}\,dx =$$

$$= \lim_{a \to +\infty} \left(\operatorname{arctg} a - \operatorname{arctg} 0 \right) + \lim_{b \to -\infty} \left(\operatorname{arctg} 0 - \operatorname{arctg} b \right)$$

$\operatorname{arctg} a \to \frac{\pi}{2}$, $\operatorname{arctg} 0 = 0$, $\operatorname{arctg} 0 = 0$, $\operatorname{arctg} b \to -\frac{\pi}{2}$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi \leftarrow$$

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f integrabile su ogni intervallo $[a, c]$, $c > a$

$f \geq 0$ $\boxed{c \to \int_a^c f(t)\,dt \text{ è crescente}}$

$c' > c$ $\quad \int_a^{c'} f = \int_a^c f + \int_c^{c'} f \geq \int_a^c f$

($\int_a^{c'} f$ e $\int_a^c f$ sono estremi inferiori; $\int_c^{c'} f \geq 0$)

Gli integrali impropri di $f \geq 0$ sono convergenti o divergenti, MAI OSCILL.

L'integrale converge se la funzione $c \to \int_a^c |f(t)|\,dt$ è limitata su $\{x > a\}$

diverge se è NON limitata.

$f, g \geq 0$ è integrabile su ogni intervallo $[a, c]$ $c > a$

$0 \leq f(x) \leq g(x) \quad \forall x \geq a$.

$$\int_a^c f \leq \int_a^c g \leq \sup \int_a^c g < +\infty$$

se $\int_a^{+\infty} g < +\infty$

$$\int_1^{+\infty} \frac{\sin^2 x}{x^2}\,dx$$

$$0 \leq \frac{\sin^2 x}{x^2} = \left|\frac{\sin^2 x}{x^2}\right| \leq \frac{1}{x^2}$$

$$\boxed{\int_1^{+\infty} \frac{1}{x^2}\,dx} = \lim_{c\to+\infty} \int_1^c \frac{1}{x^2}\,dx =$$

$$\lim_{c\to+\infty} \left[-\frac{1}{x}\right]_1^c = \lim_{c\to+\infty} \left[-\frac{1}{c} + \frac{1}{1}\right] = 1$$

$$0 \le f \le g \Rightarrow \int_a^c f \le \int_a^c g$$

Se $\int f$ diverge $\Rightarrow \int g$ diverge

Se $\int g$ converge $\Rightarrow \int f$ converge

$$\int_1^{+\infty} \frac{1}{x^p}\,dx$$

$$\int \frac{1}{x^p}\,dx = \begin{cases} \ln|x| & \text{se } p=1 \\ \dfrac{1}{1-p}\,x^{1-p} & p \neq 1 \end{cases}$$

$$\int_1^{+\infty} \frac{1}{x}\,dx = \lim_{c\to+\infty} \left[\ln c - \underbrace{\ln 1}_{0}\right] = +\infty$$

$p>1$

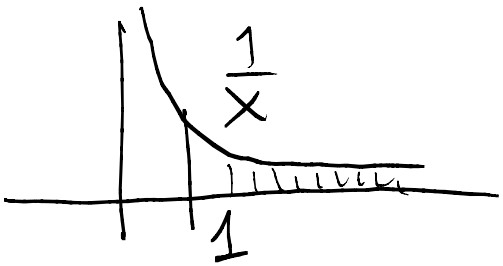
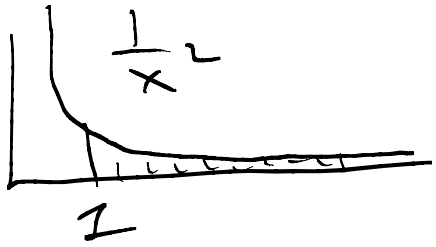
$$\int_1^{+\infty} \frac{1}{x^p}\,dx = \lim_{c\to+\infty} \frac{1}{1-p}\,x^{1-p}\Big|_1^c = \frac{1}{1-p}\left[c^{1-p} - 1\right]$$

$\lim_{c\to+\infty}$ $\quad$ 0

$p < 1 \quad c^{1-p} \to +\infty$ e l'integrale diverge.

$$\to \int_1^{+\infty} \frac{1}{x^p}\,dx \begin{cases} \text{converge} & \text{se } p > 1 \\ \text{diverge} & \text{se } p \le 1 \end{cases}$$

$$\int_1^{+\infty} \sin\frac{1}{x}\,dx \quad \text{è}$$

$$\lim_{x\to+\infty} \frac{\frac{1}{x}}{\sin\left(\frac{1}{x}\right)} = 1 \qquad \frac{1}{x} = t$$

comporta come $\int_1^{+\infty} \frac{1}{x}$ e $\sin$ quindi diverge