Note Title 12/9/2021

$$\int f(e^t)\,dt \overset{e^t = y}{=\!=} \int f(y)\,\frac{1}{y}\,dy \qquad f(x) = \frac{p(x)}{q(x)} \quad \text{p. q. p dlaczego}$$

$$t = \ln y$$

$$dt = \frac{1}{y}\,dy$$

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} =$$

$$\left.\frac{y - \frac{1}{y}}{y + \frac{1}{y}}\right|_{y=e^x} = \frac{y^2-1}{y^2+1}$$

$$y^2 - 1 \;\big|\; y^2 + 1$$

$$1$$

$$-2$$

(2) $\int f\left(x, \sqrt[n]{ax+b}\right) dx$

razionale (rapporto di polinomi in due variabili)

$\sqrt[n]{ax+b} = t$

$ax+b = t^n \qquad a \neq 0$

$x = \frac{t^n}{a} - \frac{b}{a}$

$dx = n\frac{t^{n-1}}{a}\,dt$

$= \int f\left(\frac{t^n}{a} - \frac{b}{a}, t\right) \cdot n\frac{t^{n-1}}{a}\,dt$

$b=0$

$a=1$

$n=2$

$\int \frac{\sqrt{x}\,dx}{(1+x)^2} = \int \frac{\sqrt{x}\,dx}{\left(1+(\sqrt{x})^2\right)^2}$ $\quad x>0$

$\sqrt{x} = t$

$x = t^2$

$dx = 2t\,dt$

$= \int \frac{t \cdot 2t\,dt}{(1+t^2)^2} =$

$$= \int \frac{2t^2}{(1+t^2)^2}\,dt = \int \frac{2t}{(1+t^2)^2}\cdot t =$$

$$= t\left(-\frac{1}{1+t^2}\right) + \int \frac{1}{1+t^2}\,dt =$$

$$= -\frac{t}{1+t^2} + \operatorname{arctg} t$$

$$\int \frac{2t}{(1+t^2)^2}\,dt =$$

$$= \int \frac{1}{u^2} =$$

$1+t^2 = u$

$du = 2t\,dt$

$$= -\frac{1}{u}\Big|_{u=1+t^2}$$

$$= -\frac{1}{1+t^2}$$

$$\int f(\cos t, \sin t)\,dt$$

$$\cos t = \frac{1-y^2}{1+y2} \qquad \sin t = \frac{2y}{1+y^2}$$

$$y = \operatorname{tg}\frac{t}{2} \Leftrightarrow t = 2\operatorname{arctg} y$$

$$dt = \frac{2}{1+y^2}\,dy$$

$$\int f\left(\frac{1-y^2}{1+y^2}, \frac{2y}{1+y^2}\right) \frac{2}{1+y^2}\,dy$$

$$\int \frac{1}{\sin x}\,dx = \int \frac{\frac{1+t^2}{2t}}{\frac{2}{1+t^2}}\,dt = \ln|t| = \ln\left|\operatorname{tg}\frac{x}{2}\right|$$

$t = \operatorname{tg}\frac{x}{2}$

$\operatorname{arctg} t = \frac{x}{2}$

$x = 2\operatorname{arctg} t$

$dx = \frac{2\,dt}{1+t^2}$

$t = \operatorname{tg}\frac{x}{2}$

$f(x,y) = \frac{1}{y}$

$x = \cos t$

$y = \sin t$

$f(\cos t, \sin t) = \frac{1}{\sin t}$

$$\int f(x, \sqrt{1-x^2})\,dx =$$

$x = \sin t$ oppure $x = \cos t$

$$\sqrt{1-x^2} =$$

$$= \sqrt{1-\sin^2 t} =$$

$$= |\cos t|$$

$$\int f(\sin t, |\cos t|)\cos t\,dt$$

$$=$$

$$\int \frac{x}{x+\sqrt{1-x^2}}\,dx =$$

$x = \cos t$

$dx = -\sin t\,dt$

$\sqrt{1-x^2} = |\sin t$

$$\int \frac{\cos t}{\cos t + |\sin t|}(-\sin t)\,dt$$

$f(x, \sqrt{\text{polinomio di II grado}})$

$\longrightarrow \quad \sqrt{1+x^2} \qquad \sqrt{x^2-1}$

Funzioni iperboliche

$$\cosh x = \frac{e^x + e^{-x}}{2} > 0$$

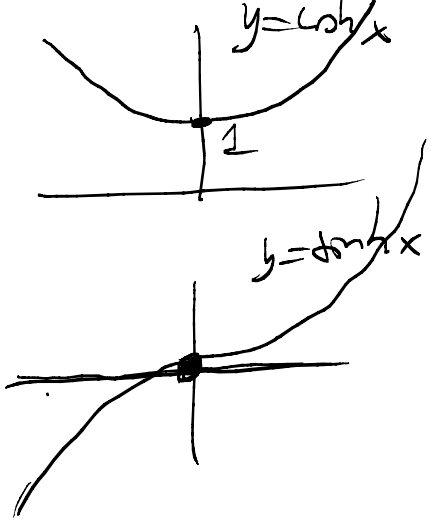
$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\boxed{\cosh^2 x - \sinh^2 x = 1}$$

$$\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 =$$

$$\begin{cases} (\cosh x)' = \sinh x \\ (\sinh x)' = \cosh x \end{cases}$$

$$= \frac{1}{4}\left[e^{2x} + e^{-2x} + 2 - \left(e^{2x} + e^{-2x} - 2\right)\right] = 1$$



$$\int \sqrt{1+x^2}\,dx \underset{\substack{x=\sinh t \\ dx=\cosh t\,dt}}{=} \int \sqrt{\underbrace{1+\sinh^2 t}_{\cosh^2 t}}\,\cosh t\,dt =$$

$\cosh t > 0$

$$= \int |\cosh t| \cosh t\,dt = \int \cosh^2 t\,dt =$$

$$= \int \left(\frac{e^t+e^{-t}}{2}\right)^2 dt = \frac{1}{4}\int e^{2t}+2+e^{-2t}\,dt$$

$$\int f(x, \sqrt{1+x^2})\,dx \underset{\substack{x=\sinh t \\ dx=\cosh t\,dt}}{=} \int f(\sinh t, \cosh t)\cosh t\,dt$$

$$g(e^t)$$

$$\int \frac{x}{x+\sqrt{1+x^2}}\,dx \overset{x=\sinh t}{=} \int \frac{\sinh t}{\sinh t+\cosh t}\cosh t\,dt = \left.\frac{x}{x+y}\right|_{y=\sqrt{1+x^2}}$$

$$= \int \frac{\frac{e^t-e^{-t}}{2}}{\frac{e^t-e^{-t}}{2}+\frac{e^t+e^{-t}}{2}} \cdot \frac{e^t+e^{-t}}{2}\,dt \overset{e^t=u}{=}$$

$$\int \sqrt{x^2-1}\,dx \overset{x=\cosh t}{=}$$

$$dx = \sinh t\,dt$$

$$\int \sqrt{\underbrace{\cosh^2 t-1}_{\sinh^2 t}}\,\sinh t\,dt = \int |\sinh t|\sinh t\,dt$$

OCCHIO AL SEGNO di $\sinh t$

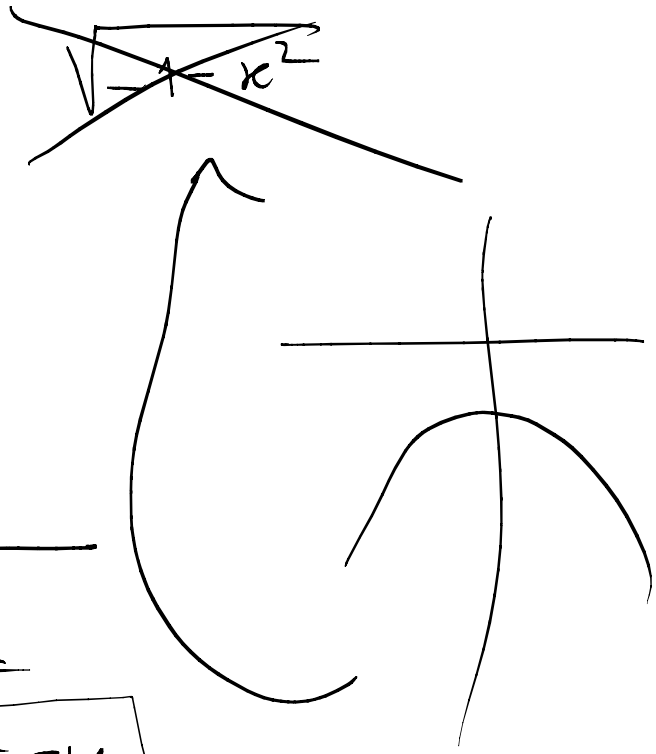
$$\int \sinh^2 t\,dt = \frac{1}{4}\int \left(e^{2t}-2+e^{-2t}\right)dt$$

$\rightarrow \sqrt{1-x^2}$, $\sqrt{x^2-1}$, $\sqrt{x^2+1}$

$x = \sin t$ o/p/m $x = \cos t$

$x = \cosh t$

$x = \sinh t$



$$\int \sqrt{x^2-4}\,dx = 2\int \sqrt{\left(\frac{x}{2}\right)^2-1}\,dx =$$

$\frac{x}{2} = y$

$dx = 2dy$

$$= 2\int \sqrt{y^2-1}\,2\,dy =$$

$$= 4\int \sqrt{y^2-1}\,dy$$

$$\int \sqrt{x^2+2x+3}\,dx =$$

$$= \int \sqrt{(x+1)^2+2}\,dx = \sqrt{2}\int \sqrt{\left(\frac{x+1}{\sqrt{2}}\right)^2+1}\,dx =$$

$$\frac{x+1}{\sqrt{2}} = t$$

$$dx = \sqrt{2}\,dt$$

$$= \sqrt{2}\cdot\sqrt{2}\int \sqrt{t^2+1}\,dt$$

$$ax^2+bx+c = a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) =$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2+\frac{c}{a}-\frac{b^2}{4a^2}\right] = a\left[\left(x+\frac{b}{2a}\right)^2-\frac{\Delta}{4a^2}\right]$$

$f(x, \sqrt{x^2+1})$ $\int f(\sqrt{x+1}, \sqrt{x-1})\,dx =$

$\sqrt{x+1} = t$

$x+1 = t^2$

$x = t^2 - 1$

$= \int f(t, \sqrt{t^2-2})\, 2t\,dt =$

$= \int f\left(t, \sqrt{2}\sqrt{\left(\frac{t}{\sqrt{2}}\right)^2 - 1}\right) 2t\,dt = \begin{vmatrix} \frac{t}{\sqrt{2}} = u \end{vmatrix}$ $u = \cosh v$.

$$\int \frac{1}{\sqrt{x}+1}\,dx = \begin{matrix}\sqrt{x}=t \\ x=t^2 \\ dx=2t\,dt\end{matrix} \int \frac{2t}{t+1}\,dt = \left[\frac{1}{y+1}\Big|_{y=\sqrt{x}}\right.$$

$$= \int \frac{2t+2}{t+1} - \frac{2}{t+1} = 2t - 2\ln|1+t|$$

$$\int tg^2 x\,dx = \begin{matrix}tgx=t \\ x=\operatorname{arctg} t \\ dx=\frac{1}{1+t^2}\,dt\end{matrix} \int \frac{t^2}{1+t^2}\,dt = t + \operatorname{arctg} t$$

$$\vdots$$

$$\int \frac{x}{1+x^2} \operatorname{tg} x \, dx = \frac{1}{2} \int \frac{2x}{1+x^2} \operatorname{tg} x \, dx =$$

$$= \frac{1}{2} \left\{ \operatorname{tg}(1+x^2) \operatorname{tg} x - \int \frac{1}{x} \operatorname{tg}(1+x^2) \, dx \right\} \text{ ---------}$$

$$\int \frac{x^2}{\sqrt{1-x^2}} dx \overset{\substack{x=\sin t \\ dx=\cos t\,dt}}{=\!=} \int \frac{\sin^2 t}{|\cos t|} \cos t \, dt =$$

$$= \operatorname{sgn}(\cos t) \int \sin^2 t \, dt =$$

$$= \operatorname{sgn}(\cos t) \int \frac{1}{2} - \frac{1}{2} \cos 2t \, dt$$

$$\cos 2t = \cos^2 t - \sin^2 t =$$
$$= 2\cos^2 t - 1 =$$
$$= 1 - 2\sin^2 t$$

$$\frac{1-\cos 2t}{2} = \sin^2 t$$

$$= \operatorname{sgn}(\cos t)\left[\frac{1}{2}t - \frac{1}{4}\sin 2t\right]$$

$$\int \sin^4 x\,dx = \int \frac{1}{4}\left[1 + \cos^2 2x - 2\cos 2x\right] =$$

$$\cos^2 t = \frac{\cos 2t + 1}{2}$$

$$\int \cos^2 2x\,dx = \frac{1}{2}\int \cos 4x + \frac{1}{2}x \qquad t = 2x \qquad \cos^2 2x = \frac{1}{2}\cos 4x + \frac{1}{2}$$

$$\frac{1}{8}\sin 4x$$

$$\int \sin^{2k+1} x\,dx = \int \underbrace{\sin^{2k} x}_{(\sin^2 x)^k} \sin x\,dx = \int \underbrace{\left(1-\cos^2 x\right)^k}_{g(\cos x)} \underbrace{\sin x\,dx}_{(\cos x)'}$$

$$\underbrace{\int \sin^{2k} x\, dx}_{I_{2k}} = \int \sin^{2k-2} x \cdot \underbrace{\sin^2 x}_{(1-\cos^2 x)}\, dx = \underbrace{\int \sin^{2k-2} x}_{I_{2k-2}} -$$

$$= \int \sin^{2k-2} x \sin x \cdot \sin x = \int \underbrace{(1-\cos^2 x)^{k-1}}_{(\sin^2 x)^{k-1}} \sin x \cdot \sin x$$

$$\to - \int \sin^{2k-2} x \cos x \cos x$$

I_{2k}

$$\int \sin^{2k-2} x \cos x = \left| \begin{matrix} y = \sin x \end{matrix} \right| = \int y^{2k-2} = \frac{1}{2k+1}$$

$$\int \sin^{2k-2} x \cos x \cos x = \cos x \frac{1}{2k+1} \sin^{2k+1} - \int (-\sin x) \frac{1}{2k+1} \sin^{2k+1}$$

$$\int_0^{\pi} \sin x \sin 2x \, dx =$$

$$= \frac{1}{2}\int_0^{\pi} (\cos x - \cos 3x)\, dx =$$

$$= \frac{1}{2}\left(\sin x - \frac{1}{3}\sin 3x\right)\Big|_0^{\pi} =$$

$$= 0$$

$\cos(x+2x) = \cos x \cos 2x - \sin x \sin 2x$

$\cos(x-2x) = \cos x \underbrace{(\cos(-2x))}_{\cos 2x} + \sin x \sin 2x$

$\cos(-x) - \cos(3x) = 2 \sin x \sin(2x)$

$\sin x \sin 2x = \frac{1}{2}[\cos x - \cos(3x)$