Note Title 12/1/2021

$$\begin{cases} \dot{x} = x \\ x(0) = 1 \end{cases}$$

$x(t)$ è la funzione incognita

equazione differenziale

condizione iniziale

Problema di Cauchy

$F = ma$

$$F(x) = m\ddot{x}$$

$x(t)$ → $\dot{x}(t)$ → $\ddot{x}$

posizione velocità accelerazione

$$x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots + a_n t^n + \dots$$

$$x'(t) = a_1 + 2a_2 t + 3a_3 t^2 + \dots$$

$$\begin{cases} x' \equiv x \\ x(0) = 1 \end{cases}$$

$t = 0 \Rightarrow a_0 = a_1$ $\qquad a_0 = 1 \Rightarrow a_1 = 1$

$$0 \equiv x' - x = (a_1 - a_0) + (2a_2 - a_1)t + (3a_3 - a_2)t^2 + \dots$$

$a_1 - a_0 = 0$

$$t\left[2a_2 - a_1 + (3a_3 - a_2)t + \dots\right]$$

$$0 \equiv t[\ ] \Rightarrow [\ ] \equiv 0 \text{ per } t \neq 0 \Rightarrow$$

$[\ ]$ continua $\Rightarrow [\ ] = 0$ anche per $t = 0$

$2a_2 - a_1 = 0$ $\Leftarrow$ $a_2 = \frac{1}{2}$ $\dots$ $a_n = \frac{1}{n!}$

$$\Rightarrow \quad x(t) = \underbrace{1}_{a_0} + \underbrace{t}_{a_1 t} + \underbrace{\frac{1}{2}t^2}_{a_2} + \underbrace{\frac{1}{3!}t^3}_{a_3} + \dots + \underbrace{\frac{1}{n!}t^n}_{} + \dots$$

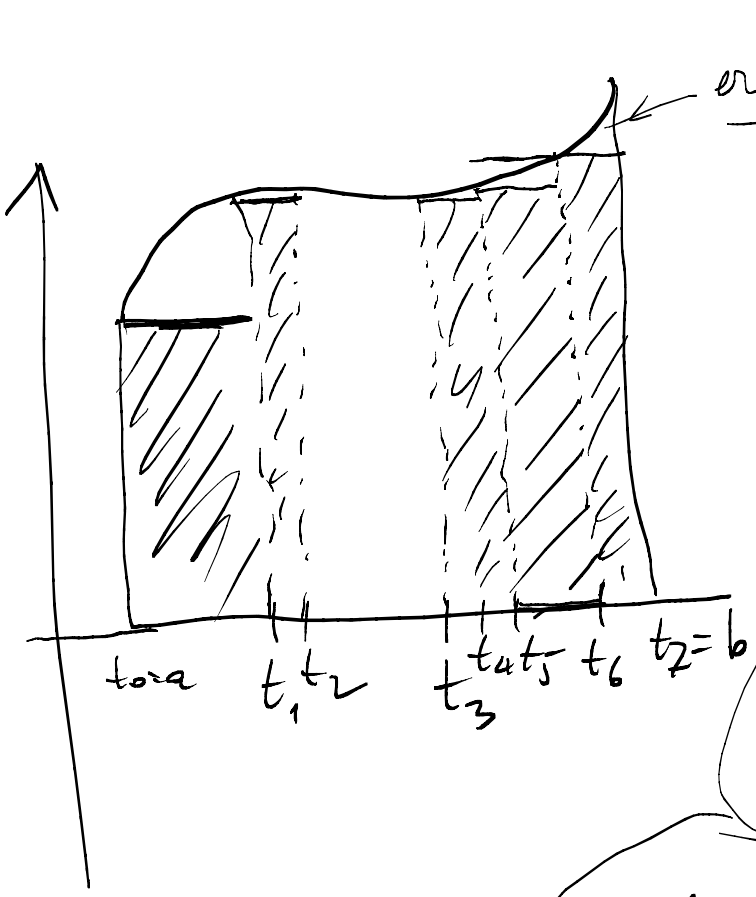
$$\boxed{\dot{x} = f(t)} \qquad x \text{ è una funzione di } f$$

$\dot{x}$: incognita; $f(t)$: nota

$$a_3 = \frac{a_2}{3} = \frac{a_{1/2}}{3} = \frac{a_4}{3!} = \frac{1}{3!}$$

INTEGRALI

$f : [a,b] \to \mathbb{R}$ limitata $\exists M : |f(x)| \le M \quad \forall x \in [a,b]$



$a = t_0 < t_1 < \dots < t_n = b$

partizione di $[a,b]$

somma inferiore (relativa a π)

stima per difetto $\longrightarrow$ $\sigma_\pi = \sum_0^{n-1} (t_{i+1} - t_i) \inf_{[t_i, t_{i+1}]} f$

somma superiore (relativa a π)

stima per eccesso $\longrightarrow$ $\Sigma_\pi = \sum_0^{n-1} (t_{i+1} - t_i) \sup_{[t_i, t_{i+1}]} f$

$a < b$

$$\overline{\int_a^b} f \equiv \inf_{\Pi} \Sigma_{\Pi}$$

Integrale SUPERIORE

al variare della partizione

$$\underline{\int_a^b} f \equiv \sup_{\Pi} \sigma_{\Pi}$$

Π partizione di $[a,b]$

f si dice integrabile in $[a,b]$ se

$$\underline{\int_a^b} f = \overline{\int_a^b} f$$

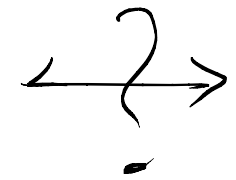
e il loro valore comune si definisce $\int_a^b f$

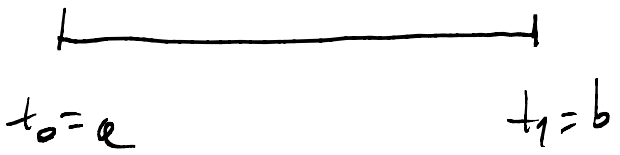
$$\sup_{\Pi} \sigma_\Pi = \inf_{\Pi} \Sigma_\Pi \left(= \int_a^b f \right) = I$$

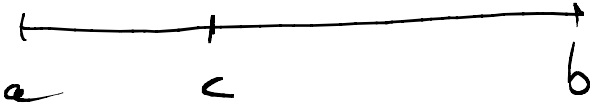
$$I = \sup_{\Pi} \sigma_\Pi \qquad \forall \varepsilon > 0 \ \exists \Pi : \ \sigma_\Pi > I - \varepsilon$$

$$\to I = \inf_{\Pi} \Sigma_\Pi \qquad \forall \varepsilon > 0 \ \exists \Pi' \quad \Sigma_{\Pi'} < I + \varepsilon$$

dicevamo $\inf_{[t_i, t_{i+1}]} f \le \sup_{[t_i, t_{i+1}]} f$

$\sigma_\Pi = (b-a) \inf_{[a,b]} f$  $(c-a) \inf_{[a,c]} f + (b-c) \inf_{[c,b]} f$





$\inf_{[a,c]} f \ge \inf_{[a,b]} f$

$\inf_{[c,b]} f \ge \inf_{[a,b]} f$

$$(c-a)\inf_{[a,c]} f + (b-c)\inf_{[c,b]} f \geq (c-a)\inf_{[a,b]} f + (b-c)\inf_{[a,b]} f =$$

somma inferiore relativa $\{a, c, b\}$

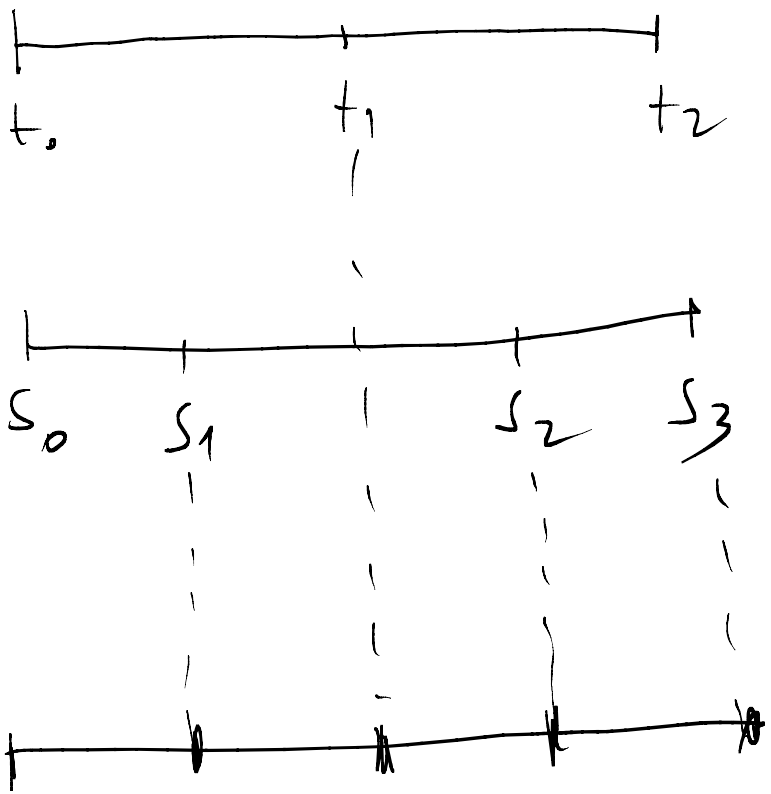
$$= (b-a)\inf_{[a,b]} f$$

somma inferiore relativa a $\{a, b\}$

$$\forall\, \pi, \pi' \Rightarrow \sigma_\pi \leq \Sigma_{\pi'}$$

Dim. $\pi'' = \pi \cup \pi'$ $\quad$ π'' è più fine di π ma di π'

$$\sigma_\pi \leq \sigma_{\pi''} \leq \Sigma_{\pi''} \leq \Sigma_{\pi'}$$



$$\overline{n} = \{t_0, t_1, t_2\}$$

$$\overline{n}' = \{s_0, s_1, s_2, s_3\}$$

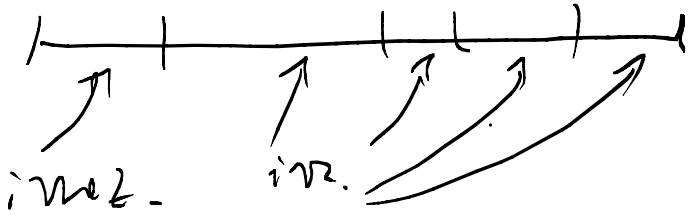
Es 1

$$D(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

$\boxed{\text{Dirichlet}}$

$$\int_0^1 D(x)$$

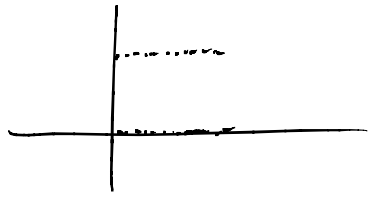
$$\sigma_\Pi = 0$$



b a perché $\inf_{[t_i, t_{i+1}]} D = 0$

$$\Sigma_\Pi = \sum (t_{i+1} - t_i) \underbrace{1}_{\sup_{[t_i, t_{i+1}]} D} = \underbrace{1-0}_{\inf \Sigma_\Pi}$$

$$\sup_\Pi \sigma_\Pi = 0$$



La funzione di Dirichlet è NON integrabile

$f(x) = 1$ integrabile

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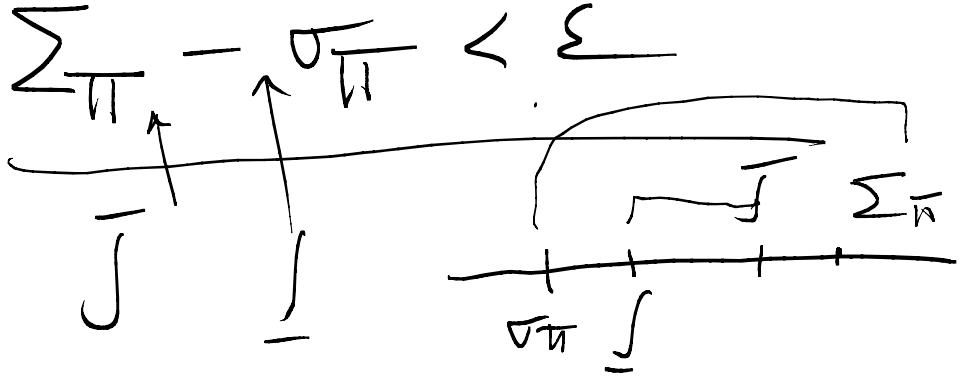
sono uguali le somme inferiori e superiori per ogni partizione.

CNS perché f sia integrabile su $[a,b]$ è che

$\forall \varepsilon > 0 \ \exists \pi$ tale che $\Sigma_\pi - \sigma_\pi < \varepsilon$



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