Note Title 10/14/2021

CONDIZIONE DI CAUCHY (prop. Cauchy)

condizione necessaria e sufficiente affinché una successione sia convergente è che

$$\forall \varepsilon > 0 \ \exists \nu : \forall n, m > \nu \quad |a_n - a_m| < \varepsilon$$

$A \Rightarrow B$

B è c.N per A

A è sufficiente per B

CN Se è convergente, $\exists L$:

$\forall \varepsilon > 0 \ \exists \nu : \forall n > \nu \quad |a_n - L| < \varepsilon$

ALLORA vale la condizione di Cauchy

DIM. $|a_n - a_m| = |a_n - L + L - a_m| \leq |a_n - L| + |L - a_m| < \varepsilon$

$|a_n - L| < \frac{\varepsilon}{2} \ \forall n > \nu$; $|L - a_m| < \frac{\varepsilon}{2} \ \forall m > \nu$ stessa ν

$\Leftrightarrow$ ha verifica la condizione di Cauchy

$\Rightarrow \exists L : a_n \to L.$

$\varepsilon = \frac{1}{2}10^{-N}$ $\quad \exists \nu_N > \nu_{N-1}$ $\quad \forall n > \nu_N$

$$a_{\nu_N+1} - \tfrac{1}{2}10^{-N} < a_n < a_{\nu_N+1} + \tfrac{1}{2}10^{-N}$$

$$a_{\nu_N+1} - \frac{10^{-N}}{2} \leq \inf_{n>\nu_N} a_n \leq a_n \leq \sup_{n>\nu_N} a_n \leq a_{\nu_N+1} + \frac{10^{-N}}{2}$$

un minorante qualunque — max dei minoranti — min dei maggioranti

$$I_N = \left[\inf_{n>\nu_N} a_n, \sup_{n>\nu_N} a_n\right]$$

$a_n \in I_N \quad \forall n > \nu_N$

$$|I_N| \leq a_{\nu_N+1} + \tfrac{1}{2}10^{-N} - a_{\nu_N+1} - \left(\tfrac{1}{2}10^{-N}\right)$$

$$= 10^{-N}$$

$$\left[\inf_{n>\nu_{N+1}} a_n \,,\, \sup_{n>\nu_{N+1}} a_n\right] \subseteq \left[\inf_{n>\nu_N} a_n \,,\, \sup_{n>\nu_N} a_n\right]$$

$n>\nu_{N+1}$ $\quad$ $n+1$

$$\inf_{n>\nu_N} a_n \le a_n \le \sup_{n>\nu_N} a_n$$

$n > \nu_{(N+1)}$

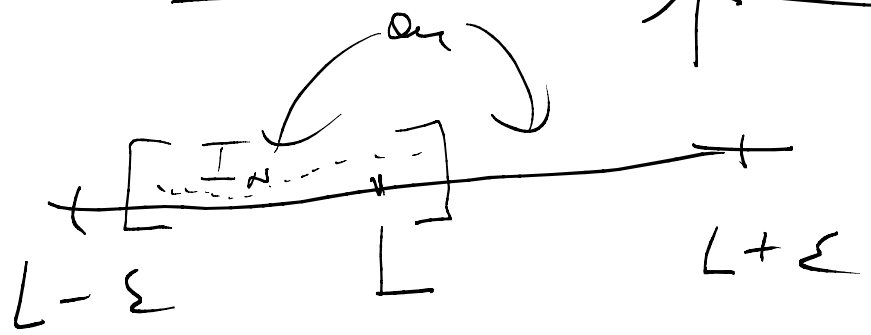
$$\inf_{n+\nu_N} \le \inf_{n>\nu_{N+1}}$$

$$\begin{cases} |I_N| = 10^{-N} \\ I_N \supseteq I_{N+1} \end{cases} \quad \forall N$$

Per il principio di Cantor

$\exists L$, unico, $L \in I_N \ \forall N$.

Fissato $\varepsilon > 0$ si N : $10^{-N} < \varepsilon$



$x \in I_N$
$L \in I_N$

$|x - L| \leq |I_N| = 10^{-N} < \varepsilon$

x e L elementi della successione contenuti in I_N

ovvero a_n, $n > \nu_N$ $\quad (a_n \in I_N \quad \forall n > \nu_N)$

UNICITA' DEL LIMITE

$$\left.\begin{array}{l} a_n \to L \\ a_n \to M \end{array}\right\} \Rightarrow L = M$$

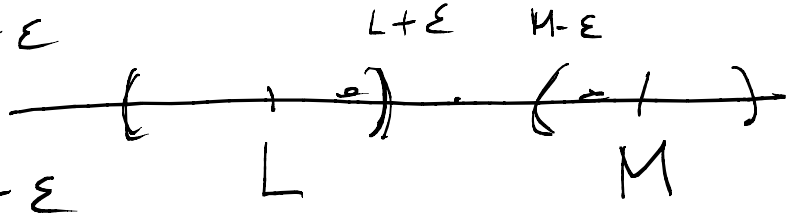
DIM

Per assurdo, supponiamo sia $M > L$

$$\varepsilon < \frac{M-L}{2}$$

$\exists\, V_1 : L-\varepsilon < a_n < L+\varepsilon$

$\exists\, V_2 \quad M-\varepsilon < a_n < M+\varepsilon$



$$\varepsilon < \frac{M-L}{2}$$

$\forall n > \{V_1, V_2\}$

$L+\varepsilon < M-\varepsilon$

$a_n < L+\varepsilon \qquad a_n > M-\varepsilon$

$\Rightarrow$

$L+\varepsilon > M-\varepsilon$

$2\varepsilon > M-L \qquad \varepsilon > \frac{M-L}{2}$

$\varepsilon < \frac{M-L}{2}$

in contraddizione con

Sottosuccessioni

$a_n = (-1)^n$

$b_n = a_{2n} = 1 \quad \forall n$

$n_k = 2k$

n_k crescente e divergente a $+\infty$

$b_k = a_{n_k}$ una sottosuccessione di a_n

$\forall n_k$ crescente e divergente a $+\infty$, se $a_n \to L$

anche $a_{n_k} \to L$

$\forall \varepsilon > 0 \ \exists \nu : \forall n > \nu \ |a_n - L| < \varepsilon$

$n_k \to +\infty$ $\forall \ \exists \bar{k} : \forall k > \bar{k} \ \ n_k > \nu$

$\Rightarrow |a_{n_k} - L| < \varepsilon$ perché $n_k > \nu$.

$n_k = 1$ k dispari

$= 2$ k pari

$$a_{n_k} = \begin{cases} a_1 & \text{k dispari} \\ a_2 & \text{k pari} \end{cases}$$

$$\overline{\Sigma} = |a_1 - a_2|$$

$(1+h)^n \geq 1+nh \qquad h \geq -1$ (con $(1+h) \geq 0$)

$n=1 \qquad 1+h \geq 1+h$ NON CI PIOVE!

occorre provare che $(1+h)^{n+1} \geq 1+(n+1)h$, supponendo vera $(1+h)^n \geq 1+nh$

$$(1+h)^{n+1} = (1+h)(1+h)^n \geq (1+h)(1+nh) =$$

$$= 1 + nh + h + nh^2 \geq 1 + h(n+1)$$

$\lim \sqrt[n]{p} = 1$

$\sqrt[n]{p} = 1+h_n \qquad h_n > 0$

$p > 1$

$p < 1$

$p = (1+h_n)^n \geq 1 + nh_n$

$0 \leq h_n \leq \frac{p-1}{n} > 0$

$0 \Rightarrow h_n \to 0$

$\Rightarrow \underline{1+h_n \to 1}$

$p = \frac{1}{(1+k_n)^n} < \frac{1}{1+nk_n}$

$p < 1$

$\sqrt[n]{p} = \frac{1}{1+k_n}$ ($\sqrt[n]{p} < 1$, $1+k_n > 1$)

$k_n > 0$

$\frac{1}{p} > 1+nk_n$

$\to 0 \le k_n < \frac{1/p - 1}{n}$

α^n $\quad 0 < \alpha < 1$ $\quad \underline{\alpha > 1}$

$\alpha = 1+h \quad \underline{h>0}$

$\underline{\alpha^n = (1+h)^n \ge 1+nh} \to +\infty$

$\alpha^n > k$

$n > \log_\alpha k$

$\alpha = \frac{1}{1+h} \quad h > 0$

$0 \le \alpha^n = \frac{1}{(1+h)^n} \le \frac{1}{1+nh} \to 0$

$a_n = \sqrt[n]{n}$ $\qquad \lim_n n^{\frac{1}{n}}$

$b_n = \sqrt{a_n} = \sqrt{\sqrt[n]{n}} = \sqrt[n]{\sqrt{n}} \Rightarrow b_n^n = \sqrt{n}$

$\sqrt{n}$: ≥ 1; $\sqrt[n]{\sqrt{n}}$: ≥ 1

$\boxed{b_n = 1 + h_n \quad h_n \geq 0}$

$\sqrt{n} = b_n^n = (1+h_n)^n \geq 1 + n h_n > n h_n \quad h_n < \frac{1}{\sqrt{n}}$

$\boxed{\frac{1}{\sqrt{n}} \; \frac{1}{n}}$ $= \frac{\sqrt{n}-1}{n} \geq h_n \geq 0$

$1 \leq a_n = b_n = (1+h_n)^2 =$

$= 1 + 2h_n + h_n^2 = 1 + \frac{2}{\sqrt{n}} + \frac{1}{n}$

$\leq 1 + \frac{3}{\sqrt{n}} \to 0$

$$\lim \frac{1}{\sqrt{n}} = 0$$

$$\left|\frac{1}{\sqrt{n}}\right| < \varepsilon$$

$$\sqrt{n} > \frac{1}{\varepsilon} \iff n > \frac{1}{\varepsilon^2}$$

$$n^{\frac{1}{n}} = e^{\frac{1}{n}\ln n} \to 0$$

$$a_n = \frac{n}{A^n}$$

$A > 1$

$\Downarrow$

$\sqrt{A} > 1$

$$\lim a_n = 0$$

$$b_n = \sqrt{a_n} = \frac{\sqrt{n}}{(\sqrt{A})^n}$$

$$\sqrt{A} = 1 + h \qquad h > 0$$

$$0 \le b_n = \frac{\sqrt{n}}{(1+h)^n} \le \frac{\sqrt{n}}{1+nh} \le \frac{\sqrt{n}}{nh} = \frac{1}{h\sqrt{n}}$$

$$0 \le a_n = b_n^2 \le \frac{1}{h^2 n} \to 0$$

$$a_n \to 0$$

$$\frac{n^2}{A^n} = \frac{n}{(\sqrt{A})^n} \cdot \frac{n}{(\sqrt{A})^n} \Longrightarrow$$

$\frac{n}{(\sqrt{A})^n} \to 0$, $\frac{n}{(\sqrt{A})^n} \to 0$

$$\frac{n^k}{A^n} = \frac{n}{(\sqrt[k]{A})^n} \cdot \dots \cdot \frac{n}{(\sqrt[k]{A})^n}$$

$\frac{n}{(\sqrt[k]{A})^n} \to 0$, $\frac{n}{(\sqrt[k]{A})^n} \to 0$

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot (n-1) \cdot n$$

Fattoriale di n

$0! \equiv 1 = 1$

$1! = 1 = 1$

$2! = 1 \cdot 2 = 2$

$3! = 1 \cdot 2 \cdot 3 = 6$

$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$

$5! = 120$

$6! = 720$

$7! = 5040$

$$\frac{2^n}{n!} = \frac{\overbrace{2 \cdot 2 \cdot \ldots \cdot 2}^{n \text{ volte}}}{1 \cdot 2 \cdots n} =$$

$$= \frac{2}{1} \cdot \left[\frac{2}{2} \cdot \frac{2}{3} \cdot \ldots \cdot \frac{2}{n-1} \right] \cdot \frac{2}{n}$$

$\frac{2}{1} \to 2$

$\left[\ldots \right]$ prodotto di fattori ≤ 1

$\frac{2}{n} \to 0$

limitato per infiniti casi.

$$\frac{n!}{n^n} = \frac{1}{n} \cdot \frac{2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n}{n \cdot n \cdot \ldots \cdot n \cdot n} \to 0$$

$\frac{1}{n} \to 0$

$\frac{2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n}{n \cdot n \cdot \ldots \cdot n \cdot n} = 1$