Note Title 10/12/2021

TEOREMI DEL CONFRONTO

Th $a_n \to L$ $b_n \to M$ $\boxed{a_n \le b_n \ \forall n \in \mathbb{N}}$

allora $L \le M$

$a_n = 0 \ \forall n \in \mathbb{N}$

$b_n = \frac{1}{n} \ \forall n \in \mathbb{N}$

$\forall \varepsilon > 0 \ \exists \nu_1 : \forall n > \nu_1$

$L - \varepsilon < a_n < L + \varepsilon$

$\boxed{a_n < b_n}$

a_n è costante e converge al suo valore $a_n \to 0$

$\forall \varepsilon \ \exists \nu_2 : \forall n > \nu_2$

$M - \varepsilon < b_n < M + \varepsilon$

$\left| \frac{1}{n} - 0 \right| < \varepsilon$

$\forall n > \max\{\nu_1, \nu_2\} \Rightarrow L - \varepsilon < a_n \le b_n < M + \varepsilon$

$\Downarrow$ $L < M + 2\varepsilon \ \forall \varepsilon > 0$

$\frac{1}{n} < \varepsilon$ $\quad n > \frac{1}{\varepsilon}$

$$\boxed{\forall \varepsilon > 0 \quad L < M + 2\varepsilon} \Leftarrow$$

se non fosse $L \leq M$ (prova per assurdo) sarebbe

$M < L$

$2\varepsilon < L - M$

Th. $a_n \to +\infty$, $b_n \geq a_n \ \forall n$ TS $\lim b_n = +\infty$

$\Downarrow$

$\forall \varepsilon > 0 \ \exists \nu : \forall n > \nu \quad a_n > \varepsilon \Rightarrow$ per lo stesso ν si ha

$$b_n \geq a_n > \varepsilon$$

$\lim 2^n = +\infty$

"$2^n \geq n+1$" $\forall n \in \mathbb{N}$

$P(1)$ $\quad 2^1 \geq 1+1$ vera

$P(n) \Rightarrow P(n+1)$

$$2^{n+1} = 2 \cdot 2^n \geq 2 \cdot (n+1) = 2n+2 \geq (n+1)+1$$

$n+2$

$-n^2 \to -\infty \qquad -n^2 < -n$

Th. confronto $\quad a_n \to L \quad c_n \to L$ (stesso limite)

$a_n \leq b_n \leq c_n \quad \forall n \in \mathbb{N}$

$\Rightarrow \lim b_n = L.$

$\forall \varepsilon > 0 \quad \exists \nu_1 \quad \forall n > \nu_1 \quad L - \varepsilon < a_n < L + \varepsilon$

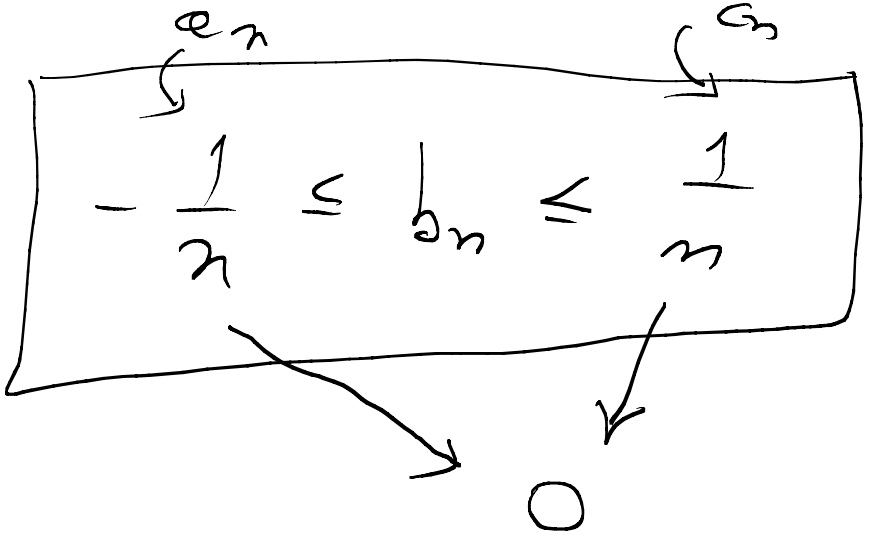
$\forall \varepsilon > 0 \quad \exists \nu_2 \quad \forall n > \nu_2 \quad L - \varepsilon < c_n < L + \varepsilon$

$\forall n > \max\{\nu_1, \nu_2\}$ dalla dopp. tesi

$$L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon$$

Es. $\boxed{b_n = \dfrac{\sin n}{n}}$

$$0 \leq |b_n| = \frac{|\sin n|}{|n|} \leq \frac{1}{n}$$



$a_n \to 0$ si dice **INFINITESIMA**

Th
Prodotto di una successione limitata per una infinitesima è infinitesima.

a_n limitata vuol dire $\exists k : |a_n| \le k$

b_n è infinitesima se $\forall \varepsilon > 0 \ \exists \nu : n > \nu \ |b_n| < \varepsilon$.

$$|a_n b_n| = |a_n||b_n| \le k|b_n| < \varepsilon$$

$|a_n|$ limitata

$k|b_n| < \varepsilon$

Scelto nella definizione di b_n il numero positivo $\frac{\varepsilon}{k}$ $\exists \nu : n > \nu \Rightarrow |b_n| < \varepsilon/k$

$a_n \to L \qquad b_n \to M \Rightarrow \qquad a_n + b_n \to L + M$

Sappiamo che $\forall \varepsilon > 0 \; \exists \nu_1 : \forall n > \nu_1 \quad |a_n - L| < \varepsilon$ $\leftarrow$

$\forall \varepsilon > 0 \; \exists \nu_2 : \forall n > \nu_2 \quad |b_n - M| < \varepsilon$ $\leftarrow$

Dobbiamo provare che $\forall \varepsilon > 0 \; \exists \nu : \forall n > \nu \; |a_n + b_n - L - M| < \varepsilon$

$$|a_n + b_n - L - M| = |(a_n - L) + (b_n - M)| \overset{\text{dis. triangolare}}{\leq}$$

$$\leq |a_n - L| + |b_n - M| < \varepsilon$$

$|a+b| \leq |a| + |b|$

$< \frac{\varepsilon}{2}$ $\qquad$ $\frac{\varepsilon}{2}$

$\forall n > \nu_1$ $\qquad$ $\forall n > \nu_2$

$\forall n > \max\{\nu_1, \nu_2\}$.

$a_n \to L \Rightarrow -a_n \to -L$

Hp $\forall \varepsilon > 0 \ \exists \nu : \forall n > \nu \ |a_n - L| < \varepsilon$

Th $\forall \varepsilon > 0 \ \exists \nu \ \forall n > \nu \ |-a_n - (-L)| < \varepsilon$

$|L - a_n| < \varepsilon$

$a_n \to L$
$b_n \to M$ $\Rightarrow$ $a_n b_n \to LM$

$|a_n b_n - LM| = |a_n b_n - L b_n + L b_n - LM| = |b_n (a_n - L) + L(b_n - M)| \leq$

$\leq |b_n| |a_n - L| + |L| |b_n - M| < \varepsilon$

$|b_n| |a_n - L|$: $|b_n| \leq k$, $|a_n - L| < \frac{\varepsilon}{2k}$ $n > \nu_1$

$|L| |b_n - M|$: $|L| < \frac{\varepsilon}{2|L|}$ $n > \nu_2$

Poiché b_n è convergente allora è limitata ed $\exists k : |b_n| \leq k \ \forall n \in \mathbb{N}$

per $n > \max\{\nu_1, \nu_2\}$

$$|a_n - L| < \frac{\varepsilon}{2k}$$

$$|b_n - M| < \frac{\varepsilon}{2|L|}$$

$a_n \to L$

$L \neq 0 \Rightarrow \frac{1}{a_n} \to \frac{1}{L}$

$$\left|\frac{1}{a_n} - \frac{1}{L}\right| = \frac{|L - a_n|}{|a_n||L|}$$

$$\big||a_n| - |L|\big| \leq |a_n - L| < \varepsilon \quad \forall n > \nu$$

$a_n \to L$

$|a_n| \to |L|$

$|a+b| \leq |a| + |b|$

$b = a + b - a$

$|b| \leq |a| + |b-a|$

$a = b + (a-b)$

$|a| \leq |b| + |a-b|$

$a_n \to L \Rightarrow |a_n| \to |L|$

$\varepsilon = \frac{|L|}{2} \quad \exists \nu : \forall n > \nu_1$

$\frac{|L|}{2} = |L| - \frac{|L|}{2} < |a_n| < |L| + \frac{|L|}{2}$

$\frac{|L - a_n|}{|a_n|\,|L|} \le \frac{|L - a_n|}{|L|\,|L/2|} \quad \forall n > \nu_1$

$\frac{|L|^2}{2} < \varepsilon$

$\varepsilon > 0 \quad |L - a_n| < \frac{\varepsilon 2}{|L|^2} \quad n > \nu_2$

$n > \max\{\nu_1, \nu_2\}$

$|b| - |a| \le |b - a|$ sono uguali

$|a| - |b| \le |a - b|$

$\big||a| - |b|\big| \le |a - b|$

$\frac{a_n}{b_n} = a_n \cdot \frac{1}{b_n}$

$a_n \to \lim a_n$

$\frac{1}{b_n} \to \frac{1}{\lim b_n}$

$$\frac{\frac{1}{n}}{\frac{1}{n^2}} \longrightarrow +\infty \qquad \frac{\frac{1}{n^2}}{\frac{1}{n}} \longrightarrow 0 \qquad \frac{\frac{1}{2n}}{\frac{1}{n}} \longrightarrow \frac{1}{2}$$

$$\frac{1}{(-1)^n n} \longrightarrow 0 \qquad \left|\frac{1}{(-1)^n n}\right| = \frac{1}{n} \longrightarrow 0$$

$a_n \to 0 \iff |a_n| \to 0$