Note Title 10/7/2021

$a+b$?

$0.149 + 0.159 \overset{n=2}{=} 0.308$

$|0.29 - (a+b)| \leq 10^{-1}$

$[a]_n \leq a \leq [a]_n + 10^{-n} \quad \forall n \in \mathbb{N}$

$[b]_n \leq b \leq [b]_n + 10^{-n} \quad \forall n \in \mathbb{N}$

$$[a]_n + [b]_n \leq a+b \leq [a]_n + [b]_n + 2\cdot 10^{-n}$$

$$a+b \in \left[[a]_n + [b]_n \, , \, [a]_n + [b]_n + 2\cdot 10^{-n} \right] = I_n$$

$x < y$

$z < w$

$x+z < y+w$

$I_n \supseteq I_{n+1} \quad \forall n \in \mathbb{N}$

$$|I_n| = [a]_n + [b]_n + 2\cdot 10^{-n} - [a]_n - [b]_n = 2\cdot 10^{-n}$$

$$2 \cdot 10^{-n} \leq 10 \cdot 10^{-n} = 10^{-n+1}$$

$\pi + e$ con 2 cifre decimali esatte

$$10^{-n+1} \leq 10^{-N}$$

$$-n+1 \leq -N$$

$$n \geq N+1$$

$$[b]_n + [a]_n \leq b + a \leq [b]_n + [a]_n + 2 \cdot 10^{-n}$$

Prodotto

$a, b > 0$

$$[a]_n \le a = [a]_n + 10^{-n}$$

$$[b]_n \le b \le [b]_n + 10^{-n}$$

$$\Downarrow$$

$$[a]_n [b]_n \le ab \le \left([a]_n + 10^{-n}\right)\left([b]_n + 10^{-n}\right) =$$

$$I_n$$

$$= [a]_n [b]_n + 10^{-n}\left([a]_n + [b]_n\right) + 10^{-2n}$$

$$|I_n| = 10^{-n}\left[[a]_n + [b]_n + 10^{-n}\right] \le 10^{-n}\left[\underbrace{[a]_0 + 1}_{[a]_n} + \underbrace{[b]_0 + 1}_{[b]_n} + \underbrace{1}_{10^{-n}}\right]$$

$$k = [a]_0 + [b]_0 + 3 \qquad \exists H : 10^H \ge k$$

$3 < 4 < 1000$

$\Rightarrow \quad |I_n| \leq K\,10^{-n} \leq 10^{H}\cdot 10^{-n} = 10^{H-n}$

ampiezza di I_n (diam.)

errore finale $\leq 10^{-N}$

$$\frac{10^{H}\cdot 10^{-n}}{10^{H-n}} \leq 10^{-N}$$

$H-n \leq -N$

$n \geq N+H$

$$[a]_n \leq a \leq [a]_n + 10^{-n} \qquad \forall n \in \mathbb{N}$$

$$-[a]_n - 10^{-n} \leq -a \leq -[a]_n$$

$\rightarrow$ I_n $\qquad |I_n| = 10^{-n}$

$a-b = a+(-b)$

$$\frac{1}{a} \in \left[\frac{1}{[a]_n + 10^{-n}} , \frac{1}{[a]_n} \right]$$

$$ax = 1 \qquad a \neq 0$$

$$\frac{e}{b} = e \cdot \frac{1}{b}$$

$$ax = b \qquad x = \frac{b}{a}$$

$x \longrightarrow ax$

$a > 0$ $\quad a^b = e^{b \lg a}$ $\quad \Leftarrow$

$x^{\frac{1}{3}} = \sqrt[3]{x}$ inverse di x^3

$\sqrt[3]{-8} = -2$

$(-8)^{\frac{1}{3}} \neq e^{\frac{1}{3}\lg(-8)}$

SUCCESSIONE = funzione il cui dominio è $\mathbb{N}$

funzioni su $\mathbb{N}$ $\quad f(n) \equiv f_n$

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a_n è una successione reale

a_n si dirà crescente se $\quad n \le m \Rightarrow a_n \le a_m$

$\forall n, m \in \mathbb{N}$

decrescente se $\quad n \le m \Rightarrow a_m \le a_n$

$\forall n, m \in \mathbb{N}$

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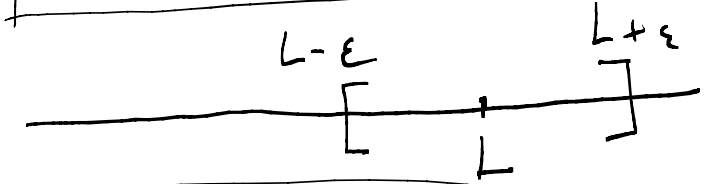
a_n si dirà convergente ad L se $\quad a_n = \frac{1}{n}$

$\forall \varepsilon > 0 \quad \exists \nu \in \mathbb{N} : \forall n > \nu \quad L - \varepsilon < a_n < L + \varepsilon$



$\varepsilon > a_n - L$

$\varepsilon > L - a_n$

$|a_n - L| < \varepsilon$

a_n converge ad L, L è il limite di a_n

$\lim_{n\to\infty} a_n = L$ oppure $\lim_n a_n = L$ oppure $\lim a_n = L$

oppure $a_n \longrightarrow L$

a_n si dirà <u>divergente</u> a $+\infty$ (modello $a_n = n$)

$\forall \varepsilon > 0 \ \exists \nu : \forall n > \nu \quad \underline{a_n > \varepsilon}$

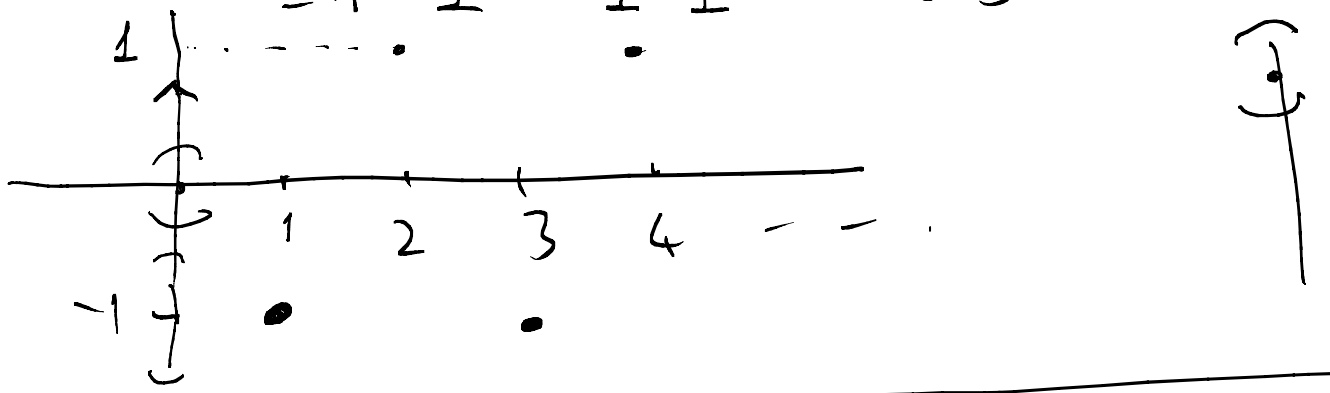
si dirà divergente a $-\infty$ $\forall \varepsilon > 0 \ \exists \nu : \forall n > \nu$ $a_n < -\varepsilon$

a_n si dirà oscillante se non è né convergente, né divergente

$a_n = (-1)^n$

$a_1 \quad a_2 \quad a_3 \quad a_4 \quad \dots \quad a_n \quad \dots$

$-1 \quad 1 \quad -1 \quad 1 \quad \dots \quad (-1)^n \quad \dots$



$a_n = (-1)^n \frac{1}{n}$

$-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5} \dots$

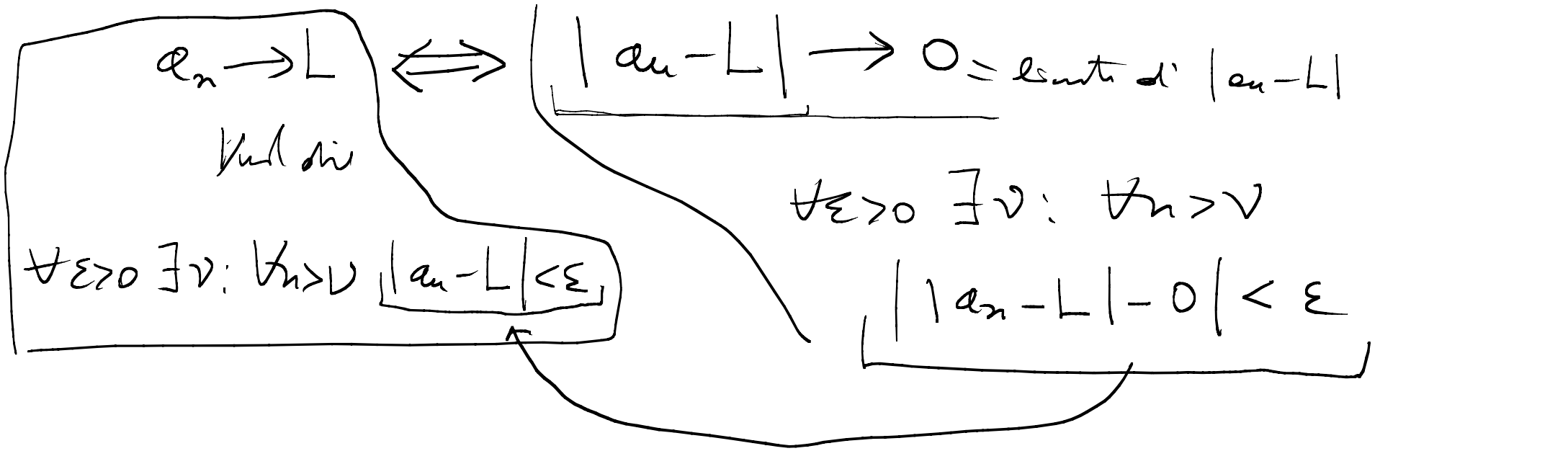
$\lim \underbrace{(-1)^n \frac{1}{n}}_{a_n} = 0$

$\forall \varepsilon > 0 \quad \exists \nu : \forall n > \nu \quad \left| (-1)^n \frac{1}{n} - 0 \right| < \varepsilon$

$\left| (-1)^n \frac{1}{n} \right| = \frac{1}{n}$

$\frac{1}{n} < \varepsilon \iff n > \frac{1}{\varepsilon}$

$\nu = \left[\frac{1}{\varepsilon} \right]_0 + 1$ = two methods per eccesso di $1/\varepsilon$



Succ. limitata a_n si dirà limitata se $\exists k \in \mathbb{R}$:

$$|a_n| \leq k \quad \forall n \in \mathbb{N}$$

$$\Updownarrow$$

$$-k \leq a_n \leq k$$

$(-1)^n$ è limitata perché

$|(-1)^n| = 1 \quad k = 1$

$-k \leq a_n \leq k$ $\forall n$ LIMITATA

$0, 1, 4, 9, \ldots, n^2$ Ciascuno INFERIORMENTE NON SUPERIORMENTE.

$a_n = 0, 2, 0, 4, 0, 6 \ldots 0, 2n, 0 \ldots$

Se fosse limitata $\exists k : |a_n| \leq k \ \forall n$

// n PARI $= 2h$
$h > 0$

$\varepsilon = 1$ $\nexists \nu : \forall n > \nu \ a_n > \varepsilon$ $2h \leq k \ \forall h$

$2h > k$ $\boxed{h > \frac{k}{2}}$

Th. a_n è convergente $\Rightarrow$ a_n è limitata

$a_n = (-1)^n$ prova che il viceversa è falso

Hp. $\exists L$ (limite) : $\leftarrow$ (Si fa che...)

$\forall \varepsilon > 0 \ \exists \nu : \forall n > \nu \quad L - \varepsilon < a_n < L + \varepsilon$

TS (da provare) $\exists K : |a_n| \leq K \ \forall n \in \mathbb{N}$

$\bar{\varepsilon} = 1 \quad \exists \bar{\nu} : \forall n > \bar{\nu} \quad L - 1 < a_n < L + 1$

è vera da $\bar{\nu}+1$ in poi

restano fuori $a_1 \dots a_{\bar{\nu}}$

$K = \max \{ |a_1|, |a_2|, \dots, |a_{\bar{\nu}}|, |L| + 1 \}$

$$\big||a_n|-|L|\big| \le \boxed{|a_n - L| < 1} \quad \varepsilon = 1 \qquad n > \bar{\nu}$$

$$-1 < |a_n| - |L| < 1$$

$$\boxed{|a_n| < 1 + |L|}$$

$$|a_n - L| \ge \big||a_n| - |L|\big|$$

$$|x-y| \ge \big||x|-|y|\big|$$

Dis. triangolare $|x+y| \le |x|+|y|$

$$|x| = |x - y + y| \le |x-y| + |y|$$

$$|y| = |y - x + x| \le \underbrace{|y-x|}_{|x-y|} + |x|$$

$$\begin{cases} |x| - |y| \le |x-y| \\ |y| - |x| \le |y-x| = |x-y| \end{cases}$$

$$|a_n| \le \max\Big\{\underbrace{|a_1|, \dots, |a_{\bar{\nu}}|}_{n \le \bar{\nu}}, \underbrace{|L|+1}_{n > \bar{\nu}}\Big\} \qquad \forall n \in \mathbb{N}$$

Th. $a_n \le a_{n+1} \quad \forall n \in \mathbb{N}$

1) se, inoltre, a_n è limitata superiormente

allora $\lim a_n = \sup \{a_n : n \in \mathbb{N}\}$

2) se, invece, non è limitata superiormente,

allora $\lim_n a_n = +\infty$

Dim

1) se $\exists k : a_n \le k \ \forall n \in \mathbb{N}$ (a_n è limitata superiormente)

$\Rightarrow$ l'insieme $\{a_n : n \in \mathbb{N}\}$ ha maggioranti (k)

e quindi $\boxed{L = \sup\{a_n : n \in \mathbb{N}\}}$ è finito (è un numero reale)

Si ha subito che $a_n \le L \ \forall n \in \mathbb{N}$ $\Rightarrow$ $\boxed{a_n < L + \varepsilon \ \forall n \in \mathbb{N}}$

Fissato $\varepsilon > 0$ ad arbitrio, dalla definizione di sup si ha che $\exists \bar{n} : a_{\bar{n}} > L - \varepsilon$

$\nu = \bar{n}$ $\quad L - \varepsilon < a_{\bar{n}} \equiv a_\nu \leq a_n \quad \forall n > \nu$

$L - \varepsilon < a_n < L + \varepsilon \quad \forall n > \nu$

Se $\{a_n : n \in \mathbb{N}\}$ NON è limitato sup.

$\forall \varepsilon > 0 \ \exists \bar{n} : \quad a_{\bar{n}} > \varepsilon$

$\nu = \bar{n} \qquad a_n \geq a_{\bar{n}} > \varepsilon \quad \Rightarrow \quad a_n > \varepsilon \ \ \forall n > \bar{n}$

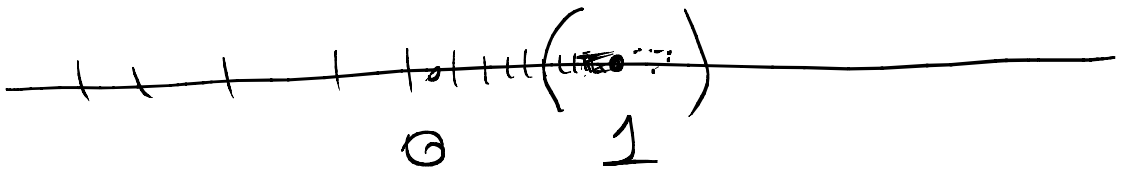
PERMANENZA DEL SEGNO

hp $a_n \to L$ $\quad L > 0$

TS $\quad \exists \nu : \forall n > \nu \quad a_n > 0$

$\forall \varepsilon > 0 \ \exists \nu : \forall n > \nu \quad L - \varepsilon < a_n < L + \varepsilon$

$\bar{\varepsilon} < L$ $\quad (\text{p.es. } \bar{\varepsilon} = \frac{L}{2})$ $\quad \exists \bar{\nu} : \forall n > \bar{\nu}$ $\quad a_n > L - \bar{\varepsilon} > 0$



$a_n = (-1)^n \frac{1}{n}$ $\quad \lim a_n = 0$

a_n è a segni alterni

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1) $a_n \le b_n \quad \forall n$

2) $a_n \to L$ $\quad \Longrightarrow \quad L \le M$

3) $b_n \to M$

$\forall \varepsilon > 0 \ \exists \nu_1 : \forall n > \nu_1 \quad L - \varepsilon < a_n < L + \varepsilon$

$\forall \varepsilon > 0 \ \exists \nu_2 : \forall n > \nu_2 \quad M - \varepsilon < b_n < M + \varepsilon$

$|a_n - L| < \varepsilon \quad a_n - \varepsilon < L < a_n + \varepsilon$

$b_n - \varepsilon < M < b_n + \varepsilon$

$a_n \le b_n \quad \forall n \Rightarrow b_n - a_n \ge 0$

Th. $a_n \ge 0 \ \forall n \Rightarrow L \ge 0$

$a_n \to L$

Dim.

$\forall \varepsilon > 0 \ \exists \nu : \forall n > \nu$

$L - \varepsilon < a_n < L + \varepsilon$

$0 \le L + \varepsilon > 0$

$\forall n > \nu$

$L + \varepsilon > 0 \quad \forall \varepsilon > 0$

$\Downarrow$

$L \ge 0$