- p polinomio non costante ($\deg p > 0$)

- $p(z_0) \neq 0$

$\Longrightarrow \exists\ \bar{z} : |p(\bar{z})| < |p(z_0)|$

$$q(w) = \frac{1}{p(z_0)} p(z_0 + w) \qquad 1)\ q(0) = \frac{p(z_0)}{p(z_0)} = 1$$

2) $\deg q = \deg p$

Se tesi vuota $\exists \bar{w} : |q(\bar{w})| < 1$

$$q(w) = 1 + a_k w^k + w^{k+1} \tilde{q}(w)$$

$$|q(w)| \le |1 + \alpha_k w^k| + |w|^{k+1} |\tilde{q}(w)|$$

sarà invariante per $|w|$ piccolo

Scegliamo w in modo da avere

$-\alpha_k w^k$ reale e negativo

$-|\alpha_k w^k| < 1$

$\Rightarrow$

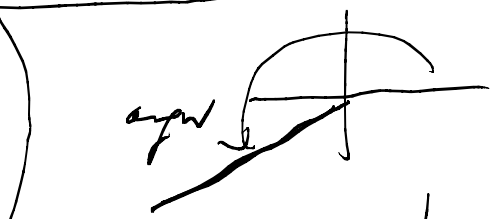
$$|1 + \alpha_k w^k| = 1 - |\alpha_k| |w|^k$$

$|a| > |b|$ $a > 0$, $b < 0$

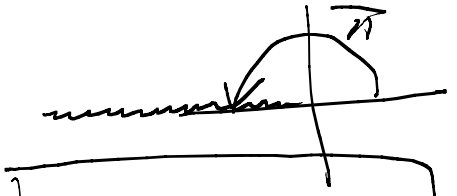
$|a+b| = |a| - |b|$

occorre che $\arg \alpha_k w^k = \pi \Rightarrow \arg \alpha_k + k \arg w = \pi$

$$\arg w = \frac{\pi - \arg \alpha_k}{k}$$



$|\alpha_k w^k| < 1$



$$|w| < \frac{1}{|\alpha_k|^{1/k}}$$

$$|q(w)| \le 1 - |\alpha_k||w|^k + |w|^{k+1}|\tilde{q}(w)| =$$

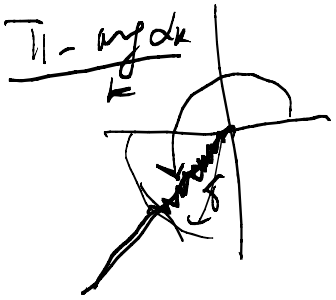
$$= 1 - |w|^k \underbrace{\left[|\alpha_k| - |w||\tilde{q}(w)| \right]}_{\to 0}$$

$|w|^k > 0$; $|\alpha_k| > 0$

$w \to 0$

$|\alpha_k| > 0$

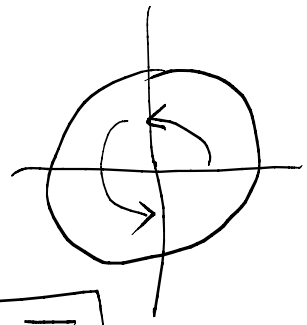
Per permanenza del segno $\left[\ \right] > 0$ se $|w| < \delta < \frac{1}{|\alpha_k|^{1/k}}$



$\Rightarrow |q(w)| < 1$

x_0 è un punto interno $\Rightarrow$ $\nabla f(x_0) = 0$

$f(x,y) = x \cos y$ $\quad [-1,1] \times [-1,2]$



$f_x = \cos y$ $\quad \nabla f = 0 \iff$ 1ª eq. $\iff y = \frac{\pi}{2}$

$f_y = -x \sin y$ $\leftarrow$ $\quad$ 2ª eq. $\Rightarrow x = 0$ $\quad \cancel{y = 0}$

Il punto critico (interno) sono: $(0, \frac{\pi}{2})$

$f(0, \frac{\pi}{2}) = 0$ $\quad f_{xx} = 0$ $\quad f_{xy} = f_{yx} = -\sin y$ $\quad f_{yy} = -x \cos y$

$$H\left(0, \frac{\pi}{2}\right) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$ è diag. con autovalori distinti

$$\begin{vmatrix} -\lambda & -1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 - 1 \lessgtr \begin{matrix} 1 \\ -1 \end{matrix}$$

Sella

NON degenere!

$$f(tx) = t^{\alpha} f(x) \qquad \underline{\alpha < 0}$$

$$f(x) = f\left(|x| \frac{x}{|x|}\right) = \underbrace{|x|^{\alpha}}_{\to +\infty} \underbrace{f\left(\frac{x}{|x|}\right)}$$

si può studiare esaminando f sulla porzione di sfera unitaria ($|x|=1$) che appartiene al dominio.

$$\inf_{\substack{|x|=1 \\ \cap \operatorname{dom} f}} f = \varepsilon > 0$$

$$f(x) > |x|^{\alpha} \varepsilon \to +\infty$$

$$\sup < \varepsilon < 0 \qquad f(x) < \varepsilon |x|^{\alpha} \to -\infty$$

$$\psi(u,v) = \phi(u_0,v_0) + \phi_u(u_0,v_0)(u-u_0) + \phi_v(u_0,v_0)(v-v_0)$$

$\phi_u(u_0,v_0)(u-u_0) + \phi_v(u_0,v_0)(v-v_0) \;\uparrow\; \alpha\phi_u + \beta\phi$

$\phi(u_0,v_0)\begin{pmatrix} u-u_0 \\ v-v_0 \end{pmatrix}$

$t=0$

$$\varphi(t) = \gamma(t_0) + t\,\dot\gamma(t_0)$$

$$\tilde\varphi(t) = \gamma(t_0) + (t-t_0)\,\dot\gamma(t_0)$$

$t=t_0$

$$x + y = -1$$

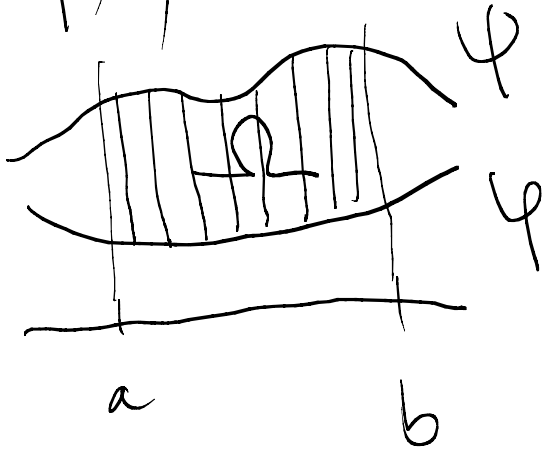
$$2x + 2y = -2$$

Superficie regolare

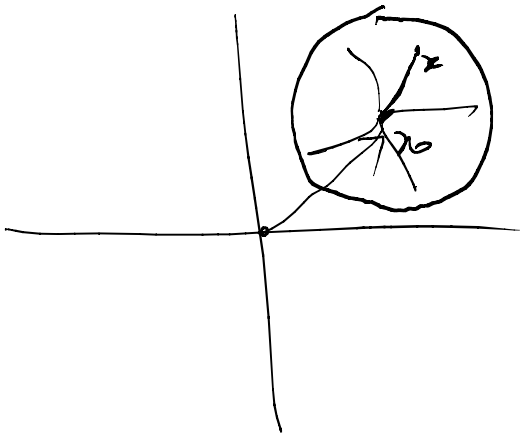
$\mathbb{R}^2$ $\quad \Omega \subseteq \mathbb{R}^2$ normale rispetto ad x

$$\Omega = \{ (x,y) \in \mathbb{R}^2 \; ; \; x \in [a,b] \; \text{e} \; \varphi(x) \le y \le \psi(x) \}$$

se $\exists\, [a,b]$, $\varphi, \psi : [a,b] \to \mathbb{R}$:



$f : \nabla f \equiv 0$ su $B_\delta(x_0) \Rightarrow f$ è costante su $B_\delta(x_0)$



$$t \to f(x_0 + t(x - x_0)) = h(t)$$

$$h(0) = f(x_0)$$

$$h(1) = f(x)$$

$\nabla f \equiv 0 \Rightarrow f$ ha derivate parziali costanti $\Rightarrow f$ è differen

da cui $h'(t) = \underbrace{\nabla f(x_0 + t(x - x_0))}_{\equiv 0} \cdot (x - x_0) \equiv 0$

h è definita su un intervallo ($[0,1]$)

$h'(t) = 0$ in $]0,1[$

$\Downarrow$

$h(t)$ è costante in $[0,1]$

$\Downarrow$

$h(0) = h(1) \iff f(x_\alpha) = f(x_\beta)$ $\forall \alpha, \beta$

$f : \mathbb{R}^2 \to \mathbb{R}$ $\quad$ f continua dappertutto

$$\boxed{\lim_{\infty} f = \pm\infty}$$

$\Downarrow$

f ha minimo.

$x_0 \in \mathbb{R}^2$ arbitrario tale che $|f(x_0)| \neq 0$ $\quad$ $|f(x_0)| = \varepsilon$

$\forall \varepsilon > 0$ $\quad$ $\exists \delta : |x| > \delta$ $\quad$ $|f(x)| > \varepsilon$ $\quad$ $|x| > \delta$

$\varepsilon = 1$

$\{ x \in \mathbb{R}^2 : |f(x)| \le \varepsilon \} \subseteq B_\delta(0)$

limitato

chiuso perché $x_n \to \bar{x}$: $|f(x_n)| \le \varepsilon \Rightarrow$

$|f(\bar{x})| \le \varepsilon$

f ha minimo in x^* $\quad B_\delta(0) \cap \mathrm{dom} f$ (contiene un chiuso limitato)

$\forall x \in \mathrm{dom} f$

- se $x \in B_\delta(0)$
- $x \notin B_\delta(0) \Rightarrow f(x) > \varepsilon$

$x_0 \in B_\delta(0)$

$f(x^*) \le f(x_0) < f(x)$

$A \in C^0$ integrabile $\Rightarrow$ $\int_\gamma A$ dipende solo dagli estremi

$$\int_\gamma A = \int_a^b A(\gamma(t))\,\dot\gamma(t)\,dt \xrightarrow[\exists f:\nabla f = A]{A \text{ integr.}} \int_a^b \nabla f(\gamma(t))\,\dot\gamma(t)\,dt =$$

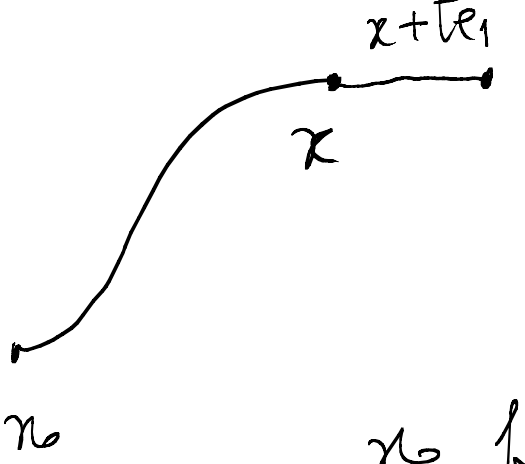
a // $\nabla f = A \in C^0 \Rightarrow f$ di classe

$\frac{d}{dt}\left[f(\gamma(t))\right]$

$$= \int_a^b \frac{d}{dt}\left[f(\gamma(t))\right] \underset{\text{TdNC}}{=\!=} f(\gamma(b)) - f(\gamma(a))$$

$\in C^0$

dipende solo da $\gamma(a)$ e $\gamma(b)$



$t \in [0,h]$

$$\boxed{f(x) \equiv \int_{\gamma_{\overrightarrow{x_0 x}}} A}$$

In Analisi I il potenziale è $\int_a^x f(t)\,dt$

x_0 fisso

x variabile

$$\frac{f(x+he_1)-f(x)}{h} =$$

Per provare che $\nabla f \equiv A$

$$= \boxed{\frac{1}{h}\left[\int_0^h \underbrace{A(x+te_1)\cdot e_1}_{A_1(x+te_1)}\,dt\right]} = A_1(x+\xi e_1)$$

media integrale di una f continua

$\xi \in [0,h]$ $\quad h \to 0 \Rightarrow \xi \to 0$

e, essendo A continua $\lim_{h\to 0} A_1(\overbrace{x+\xi e_1}^{x}) =$

$= A_1(x)$

$$-\frac{y}{\sqrt{x^2+y^2}}\,dx + \frac{x}{\sqrt{x^2+y^2}}\,dy \qquad \text{dom } \mathbb{R}^2 \setminus \{0\}$$

$$\begin{cases} f_x = -\dfrac{y}{\sqrt{x^2+y^2}} \\ f_y = \dfrac{x}{\sqrt{x^2+y^2}} \end{cases}$$

$$\frac{\partial\left(-\frac{y}{\sqrt{x^2+y^2}}\right)}{\partial y} =$$

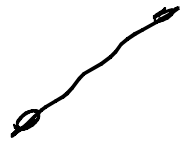
$$= \frac{-\sqrt{x^2+y^2} + y\,\frac{2y}{2\sqrt{x^2+y^2}}}{x^2+y^2} =$$

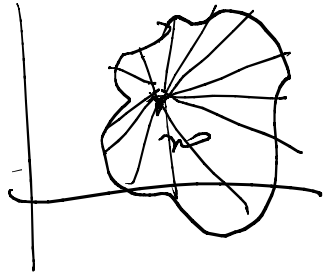
$$= \frac{-\sqrt{x^2+y^2} + \frac{y^2}{\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{-x^2-y^2+y^2}{(x^2+y^2)^{3/2}}$$

$$\sqrt{x^2+y^2}$$

$$\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right) = \frac{\sqrt{x^2+y^2} - x\,\frac{2x}{2\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{x^2+y^2-x^2}{(x^2+y^2)\sqrt{x^2+y^2}}$$

Ω stella rispetto a x_0





CAMPI

I) Caso $x_0 = 0$

$$h(\lambda, t) \equiv \gamma(t)\,(1-\lambda)$$

$\in$ segmento di estremi 0 e $\gamma(t)$

e tutti appartengono a Ω

$$h(0,t) = \gamma(t)$$

$$h(1,t) \equiv 0$$

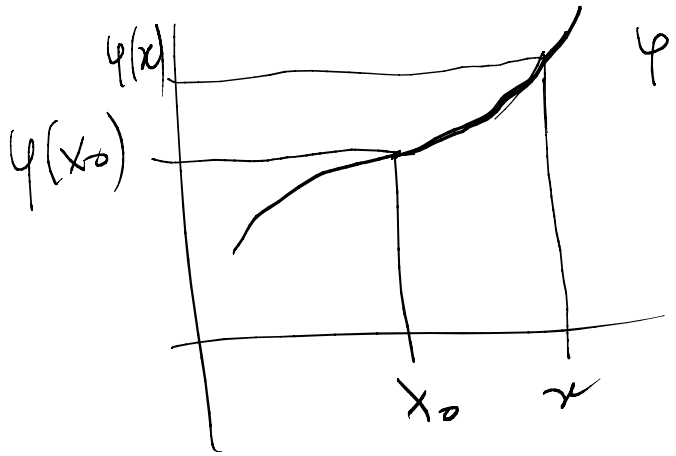
h continua

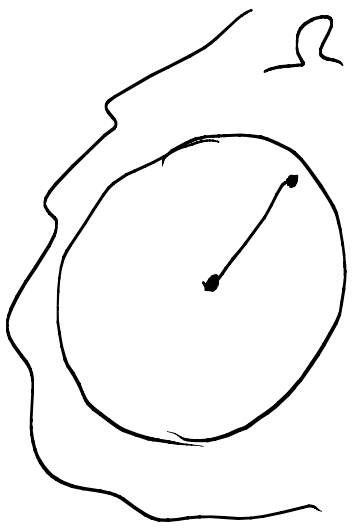
II) $x_0 \neq 0$

$$h(\lambda, t) = x_0 + (1-\lambda)\,[\gamma(t) - x_0]$$

$\in$ segmento di estremi x_0 e $\gamma(t)$

$$\varphi'(x_0) = \lim_{x \to x_0} \frac{\varphi(x) - \varphi(x_0)}{x - x_0}$$





$$\underbrace{f(x, \varphi(x))}_{=0} - \underbrace{f(x_0, \varphi(x_0))}_{=0} =$$

$$= f_x\Big(x_0 + \zeta(x - x_0), \varphi(x_0) + \zeta\big(\varphi(x) - \varphi(x_0)\big)\Big)(x - x_0) +$$

$$+ f_y(\quad)\big(\varphi(x) - \varphi(x_0)\big)$$

> 0

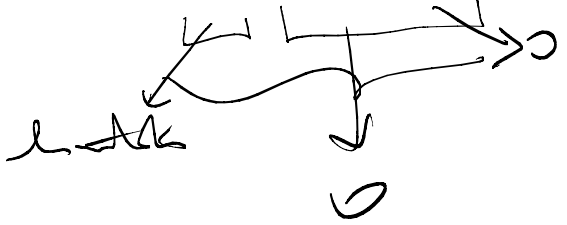
$\zeta \in [0, 1]$

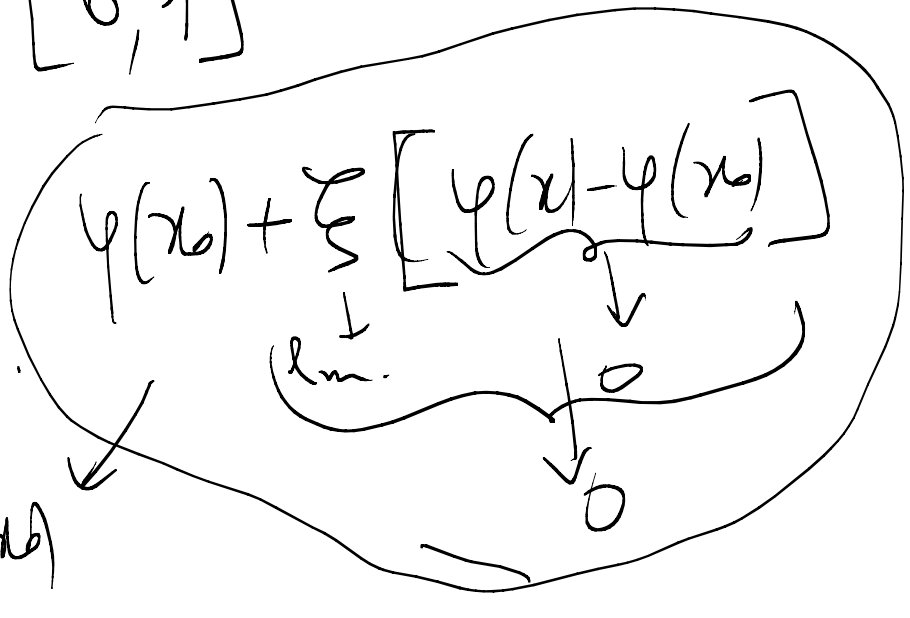
$$\frac{\varphi(x) - \varphi(x_0)}{x - x_0} = - \frac{f_x(\quad)}{f_y(\quad)}$$

quando $x \to x_0 \Rightarrow \varphi(x) \to \varphi(x_0)$

$\Rightarrow \xi \in [0,1]$

$x_0 + \xi(x - x_0) \to x_0$





$$\frac{\varphi(x) - \varphi(x_0)}{x - x_0} \to -\frac{f_x(x_0, y_0)}{f_y(x_0, y_0)}$$