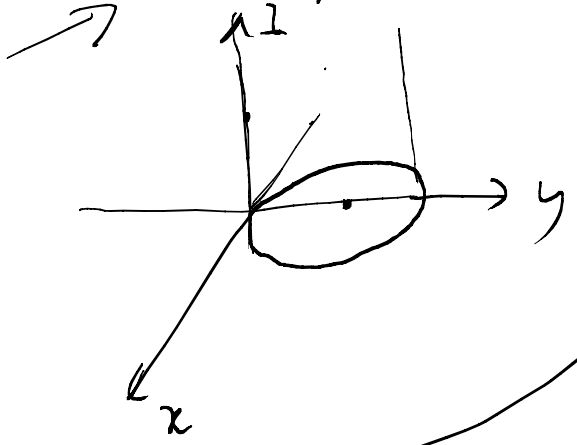
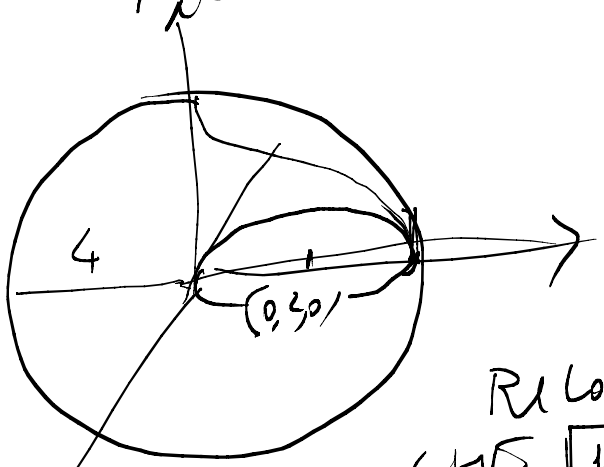
Note Title 6/3/2021

$$\{z \geq 0\} \cap \{x^2+y^2 \leq 4y\} \cap \{x^2+y^2+z^2 = 16\}$$

$x^2 + (y-2)^2 \leq 4$



sfera di centro $(0,0,0)$ e raggio 4



RICORDARE CHE $\sqrt{16-16\sin^2\theta} = 4|\cos\theta|$

$$z = \sqrt{16-x^2-y^2}$$

$$\int_{x^2+y^2 \leq 4y} \sqrt{16-x^2-y^2}\, dx\,dy$$

POLARI PIANE

$$\rho^2 < 4\rho\sin\theta \Rightarrow \rho < 4\sin\theta$$

$\rho > 0$

$\sin\theta \geq 0$

$\theta \in [0, \pi]$

$$\int_0^{\pi} d\theta \int_0^{4\sin\theta} \rho\sqrt{16-\rho^2}\, d\rho$$

$16-\rho^2 = t$

caso "scalare"

(x_0, y_0) interno ad Ω

$f(x,y)$ continua in Ω

$f(x_0, y_0) = 0$ ⟵

$f_y(x_0, y_0) > 0$ $\Longrightarrow$ $\det \frac{\partial f}{\partial y} \neq 0$

$\det \frac{\partial (f_1 \cdots f_m)}{\partial (y_1 \cdots y_m)} (x_0, y_0) = \begin{vmatrix} \frac{\partial f_1}{\partial y_1}(x_0,y_0) & \cdots & \frac{\partial f_1}{\partial y_m}(x_0,y_0) \\ \vdots & & \\ \frac{\partial f_m}{\partial y_1}(x_0,y_0) & \cdots & \frac{\partial f_m}{\partial y_m}(x_0,y_0) \end{vmatrix} \neq 0$

$x_0 \in \mathbb{R}^k \quad y_0 \in \mathbb{R}^m$

$f : \Omega \to \mathbb{R}^m \quad \Omega \subseteq \mathbb{R}^{k+m}$

$\varphi'(x) = -\frac{f_x(x, \varphi(x))}{f_y(x, \varphi(x))}$ in una variabile scalare

$\exists\, \delta \quad \varphi : B_\delta(x_0) \to \mathbb{R}^m$ $\quad \subseteq \mathbb{R}^k$

$\varphi(x_0) = y_0$

$f(x, \varphi(x)) \equiv 0$ $\quad \forall x \in B_\delta(x_0)$

$\frac{\partial (\varphi_1 \cdots \varphi_m)}{\partial (x_1 \cdots x_k)}(x) = -\left[\frac{\partial (f_1 \cdots f_m)}{\partial (y_1 \cdots y_m)}\right]^{-1} \frac{\partial (f_1, \ldots, f_m)}{\partial (x_1, \ldots x_k)} (x, \varphi(x))$

$m \times m$ $\qquad$ $m \times k$

$y_i = \varphi_i(x)$ risolve il sistema $m \times k$ rispetto a y_i

Area cono $x^2+y^2 = 3z^2$

$z \le 0$

$x^2+y^2 = 4y$

$$z = -\frac{1}{\sqrt{3}}\sqrt{x^2+y^2} = f(x,y)$$

$$\int_{dom} \sqrt{1+f_x^2+f_y^2}\,dx\,dy$$

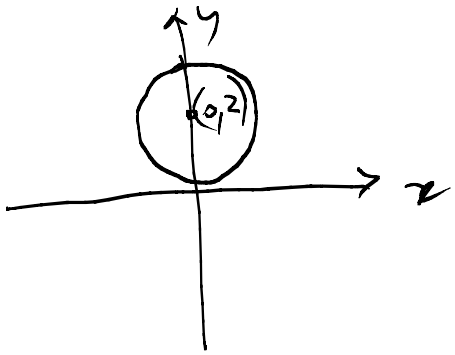
$$\int_{x^2+y^2\le 4y} \sqrt{1+\underbrace{\frac{1}{3}\frac{x^2}{x^2+y^2}}_{f_x^2}+\underbrace{\frac{1}{3}\frac{y^2}{x^2+y^2}}_{f_y^2}}\,dx\,dy =$$

$$f_x = -\frac{1}{\sqrt{3}}\frac{2x}{2\sqrt{x^2+y^2}}$$

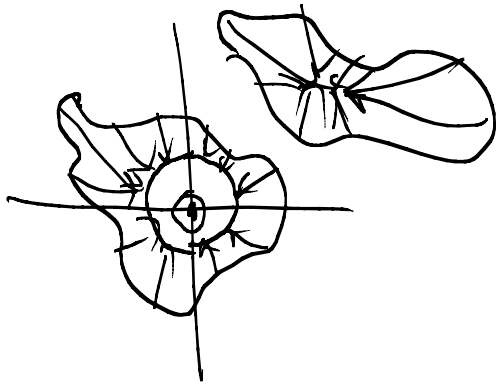
$$f_y = -\frac{1}{\sqrt{3}}\frac{y}{\sqrt{x^2+y^2}}$$

$$= \int_{x^2+y^2\le 4y} \sqrt{\frac{3x^2+3y^2+x^2+y^2}{3(x^2+y^2)}}\,dx\,dy$$

$$= \int_{x^2+y^2 \le 4y} \sqrt{\frac{4}{3}} \, dx\,dy =$$



$$= \frac{2}{\sqrt{3}} \pi \cdot \underset{r^2}{4} = \frac{8}{\sqrt{3}} \pi$$

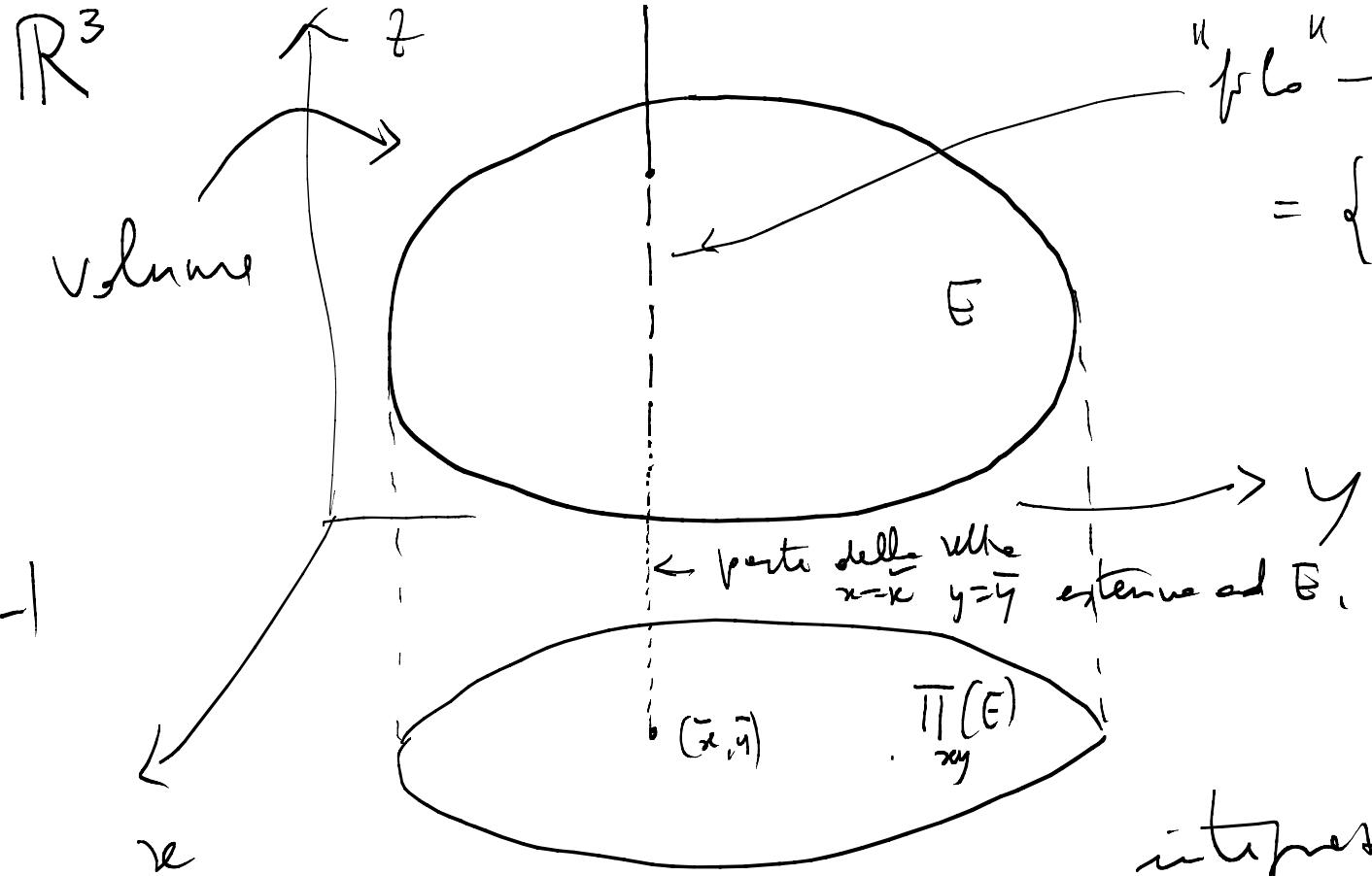


$\cos^3\theta$

$2\cos^2\theta - 1 = \cos 2\theta$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$\cos^4\theta = \frac{1}{4}(1+\cos^2 2\theta + 2\cos 2\theta)$

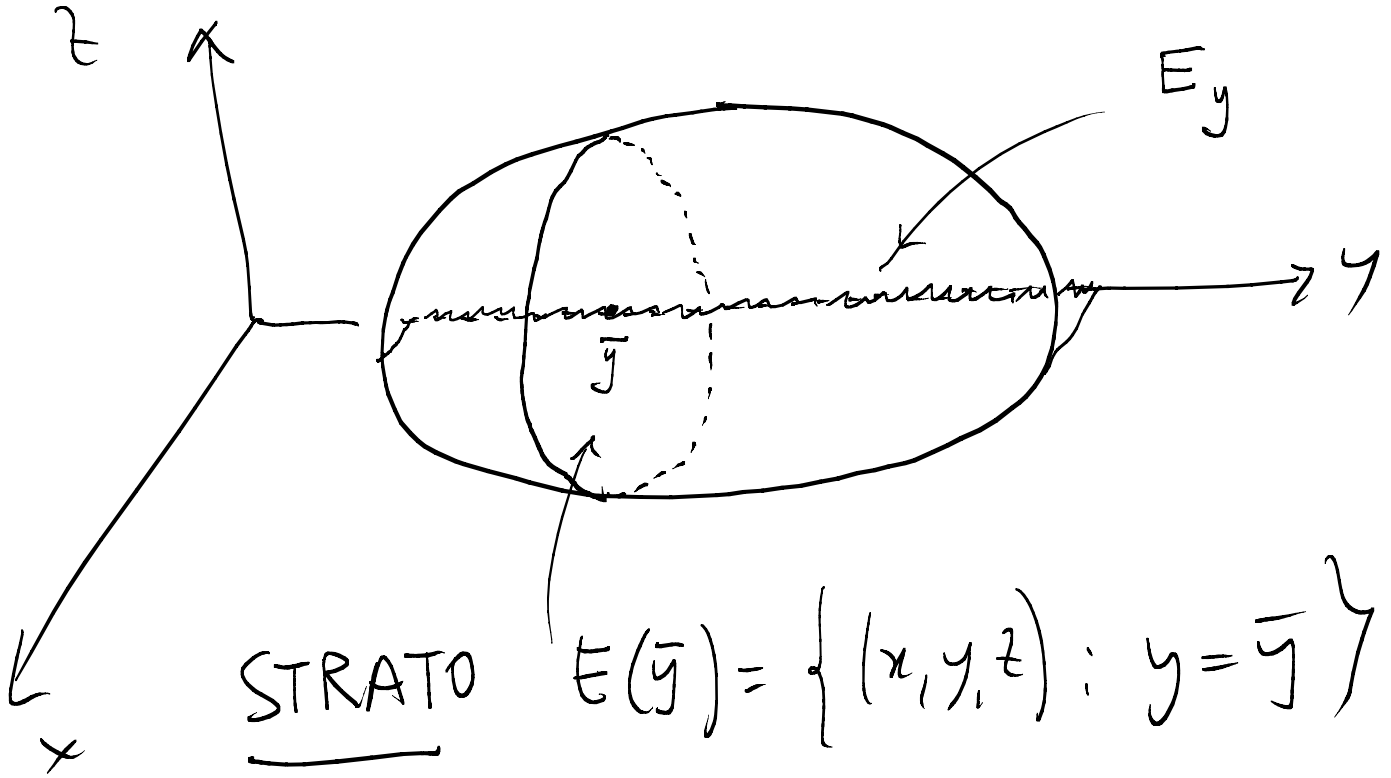


"filo" $\longrightarrow E(\bar{x},\bar{y}) = \{(x,y,z) \in E \quad \begin{matrix} x=\bar{x} \\ y=\bar{y} \end{matrix}\}$

FILI

← parte della retta $x=\bar{x}$ $y=\bar{y}$ esterna ad E.

$$\int_E f(x,y,z)\,dx\,dy\,dz = \overbrace{\int_{E_{x,y}} dx\,dy \underbrace{\int_{E(x,y)} f(x,y,z)\,dz}_{\text{filo}}}^{\text{integrazione per fili}}$$



STRATO $E(\bar{y}) = \{(x,y,z) : y = \bar{y}\}$ (sezione)

$$\int_E f(x,y,z)\,dx\,dy\,dz = \int_{E_y = \Pi_y(E)} dy \int_{E(y)} f(x,y,z)\,dx\,dz$$

↑ strato

$$M = \frac{1}{b-a} \int_a^b f(t)\,dt$$ MEDIA INTEGRALE

f integrabile dice che $M \in \left[\inf_{[a,b]} f, \sup_{[a,b]} f \right]$

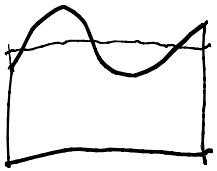
f continua $\Longrightarrow$ ogni $M \in \left] \inf_{[a,b]} f, \sup_{[a,b]} f \right[$

è assunto da f (th. valori intermedi)

$\exists\, \xi \in [a,b]$: $\boxed{M = f(\xi)}$

$\frac{1}{N} \sum_{i}^{N} a_n$

$f > 0$



$$\nabla |x| = \frac{x}{|x|} = \left(\frac{2x_1}{2\sqrt{\sum x_i^2}}, \frac{2x_2}{2\sqrt{\sum x_i^2}}, \dots, \frac{2x_n}{2\sqrt{\sum x_i^2}} \right)$$

x vettore

$|x|$ scalare

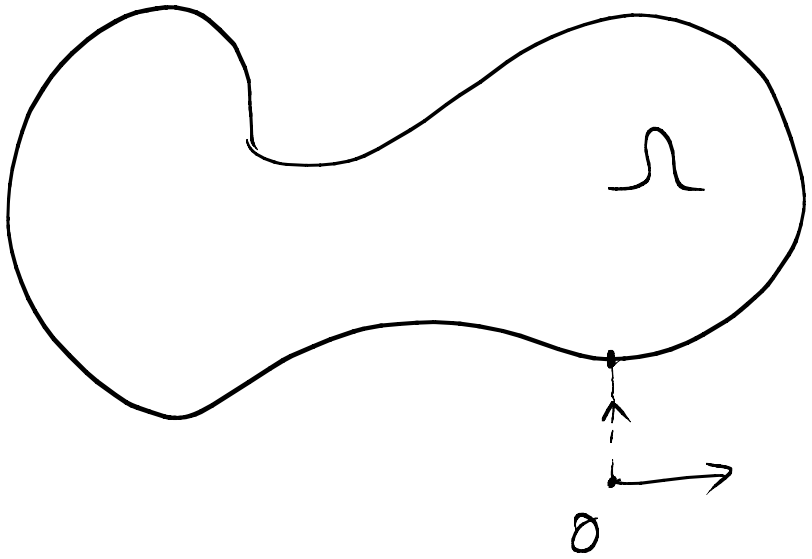
$x\,dy - y\,dx$ **rotore**

$-y\,dx + x\,dy \approx (-y, x)$

CAMPI E FORME (2)

$$\frac{\partial(-y)}{\partial(y)} = -1 \neq \frac{\partial x}{\partial x} = 1$$

non rispetta la condizione del rotore $\Rightarrow$ **NON È INTEGRABILE**

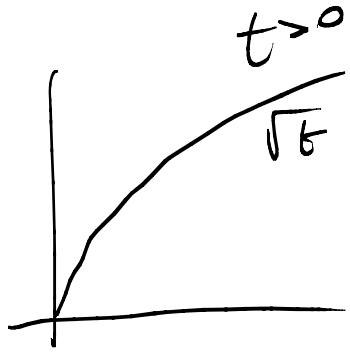


$$\min_{\Omega} \sqrt{x^2+y^2}$$

>0

$$\approx$$

$$\min (x^2+y^2)$$



$w_1 \dots w_k$ base di $X \cap Y$ $\dim X \cap Y = k$

$w_1 \dots w_k x_{k+1} \dots x_n$ base di X $(\dim X = n)$

$w_1 \dots w_k y_{k+1} \dots y_m$ base di Y $(\dim Y = m)$

occorre provare che $\dim(X+Y) = m+n-k$

Per fare ciò si prova che $w_1 \dots w_k x_{k+1} \dots x_n y_{k+1} \dots y_m$ è una base di $X+Y$

$X+Y \subseteq \langle \; \rangle$

Per provare l'indip. lineare

$$\sum_{1}^{k} \alpha_i w_i + \sum_{k+1}^{n} \beta_j x_j + \sum_{k+1}^{m} \gamma_h y_h = 0$$

$$\Rightarrow \alpha, \beta, \gamma = 0 \text{ tutti?}$$

$$n + x + y = 0$$

$w + x = -y \Rightarrow y \in X \cap Y$

$w \in X$, $-y \in Y$

$y \in Y$ per def. $\sum_{k+1}^{n} \gamma_h y_h$

ma anche $y \in X$

$(w+y)+x = 0$

$w+y \in X \cap Y$ $\quad x \in \langle x_{k+1}, \dots x_n \rangle$

$X \cap Y = \langle w_1 \dots w_k \rangle$

per il lemma di scambio.

della base $w_1 \dots w_k, x_{k+1} \dots x_n$

la somma di $X \cap Y$ e $\langle x_{k+1} \dots x_n \rangle$

è diretta e quindi

$(w+y)+x = 0 \Rightarrow x = 0$ $\quad w+y = 0$

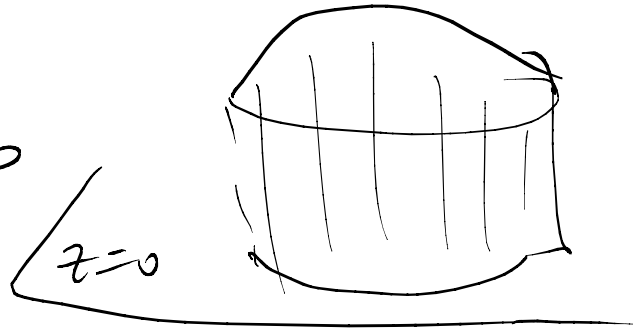
$x = 0 \Rightarrow \sum_{k+1}^{n} \beta_j x_j = 0$ ed essendo $x_{k+1} \dots x_n$ indip $\Rightarrow \beta_j = 0 \quad \forall j = k+1 \dots n$.

$\sum_{1}^{k} \alpha_i w_i + \sum_{k+1}^{n} \gamma_h y_h = 0$ e poiché w_i, y_h sono una base di $Y \Rightarrow$

$\alpha_i = 0 \quad \forall i = 1 \dots k \qquad \gamma_h = 0 \quad \forall h = k+1 \dots m$

$$\{z \geq 0\} \cap \{x^2+y^2 \leq 4y\} \cap \{x^2+y^2+z^2 = 16\} = E$$

$$z = \sqrt{16 - x^2 - y^2} \geq 0$$



$$|E| = \int_{x^2+y^2 \leq 4y} \sqrt{16-x^2-y^2}\, dx\,dy = \int_0^{\pi} d\theta \int_0^{4\sin\theta} d\rho\, \rho \sqrt{16-\rho^2}$$

$x^2+y^2 \leq 4y$

$\rho^2 \leq 4\rho\sin\theta \overset{\rho \geq 0}{\Longrightarrow}$ $\boxed{\rho \leq 4\sin\theta}$

$\sin\theta \geq 0 \Rightarrow \theta \in [0, \pi]$

$$= \int_0^{\pi} d\theta \int_0^{4\sin\theta} d\rho\, \rho \sqrt{16-\rho^2} = -\frac{1}{2}\int_0^{\pi} d\theta \int_0^{4\sin\theta} \sqrt{16-\rho^2}\, \underbrace{(-2\rho)\, d\rho}_{dt} \overset{16-\rho^2=t}{=}$$

$$= -\frac{1}{2}\int_0^{\pi} d\theta \; \frac{2}{3}(16-\rho^2)^{\frac{3}{2}} \Big|_0^{4\sin\theta} =$$

$$\sqrt{a^2} = |a|$$

$$= -\frac{1}{3}\int_0^{\pi} d\theta \left(16-16\sin^2\theta\right)^{\frac{3}{2}} = -\frac{1}{3}\int_0^{\pi} d\theta \left(16\cos^2\theta\right)^{\frac{3}{2}} =$$

$$= -\frac{1}{3}\int_0^{\pi} d\theta \left[4|\cos\theta|\right]^3 = -\frac{1}{3}\int_0^{\frac{\pi}{2}} 4\cos^3\theta - \int_{\frac{\pi}{2}}^{\pi} -4\cos^3\theta$$

Th GRAM

$u_1 \dots u_n$ independent

$$\begin{pmatrix} u_1 u_1 & u_1 u_2 & \cdots & u_1 u_n \\ u_2 u_1 & u_2 u_2 & & u_2 u_n \\ \vdots & & & \\ u_n u_1 & u_n u_2 & & u_n u_n \end{pmatrix}$$

same independent

$0 = \sum \lambda_i$ column $\Rightarrow \lambda_i = 0 \ \forall i$

$$0 = \begin{pmatrix} u_1 \lambda_1 u_1 \\ u_2 \lambda_1 u_1 \\ \vdots \\ u_n \lambda_1 u_1 \end{pmatrix} + \begin{pmatrix} u_1 \lambda_2 u_2 \\ u_2 \lambda_2 u_2 \\ \vdots \\ u_n \lambda_2 u_2 \end{pmatrix} + \dots + \begin{pmatrix} u_1 \lambda_n u_n \\ u_2 \lambda_n u_n \\ \vdots \\ u_n \lambda_n u_n \end{pmatrix} =$$

$$= \begin{pmatrix} u_1 \sum_1^n \lambda_i u_i \\ u_2 \sum_1^n \lambda_i u_i \\ \vdots \\ u_n \sum_1^n \lambda_i u_i \end{pmatrix} \Leftrightarrow \begin{cases} u_1 \sum_1^n \lambda_i u_i = 0 \\ \vdots \\ u_n \sum_1^n \lambda_i u_i = 0 \end{cases}$$

Proviamo in span di vettori indip.

moltiplicando la i-esima equazione per λ_i e sommando membro a membro (mettendo in evidenza il fattore comune $\sum_1^n \lambda_i u_i$) si ottiene

$$\left| \sum_1^n \lambda_i u_i \right|^2 = \left(\sum_1^n \lambda_i u_i \right) \left(\sum_1^n \lambda_i u_i \right) = 0 \Rightarrow \sum_1^n \lambda_i u_i = 0 \Rightarrow \lambda_i = 0 \ \forall i$$

$\hookrightarrow$ indipendenti

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{|(x,y)|}}{\sqrt{x^2+y^2}} \geq \qquad |(x,y)| = \sqrt{x^2+y^2}$$

$$\geq \lim_{(0,0)} \frac{[x^2+y^2]^{1/4}}{[x^2+y^2]^{1/2}} \overset{(x,y)\neq(0,0)}{=} \lim_{(0,0)} [x^2+y^2]^{-\frac{1}{4}} = +\infty$$

$$f(x,y) = x^3+y^3-1$$

livello critico (contenente $(0,0)$) $\quad f=-1$

$$\nabla f(x,y) = \begin{pmatrix} 3x^2 \\ 3y^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ solo per } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \notin \{f(x,y)=0\}$$

$$x^3 + y^3 - 1 = 0$$

$y = 0 \Rightarrow x = 1$ $\qquad (1,0) \in \{ f(x,y) = 0 \} \neq \emptyset$

e non contiene punti critici

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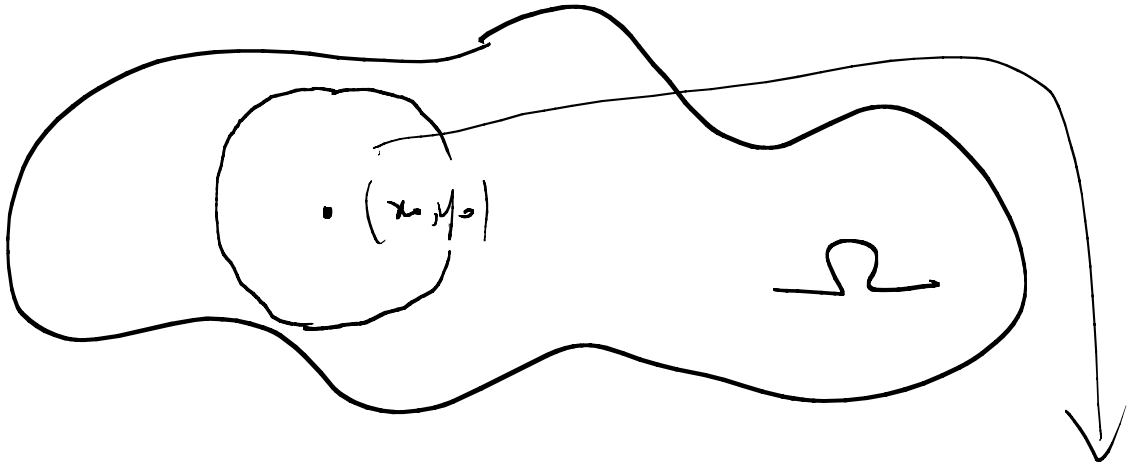
x_0 è critico per f se $\nabla f(x_0) = 0$

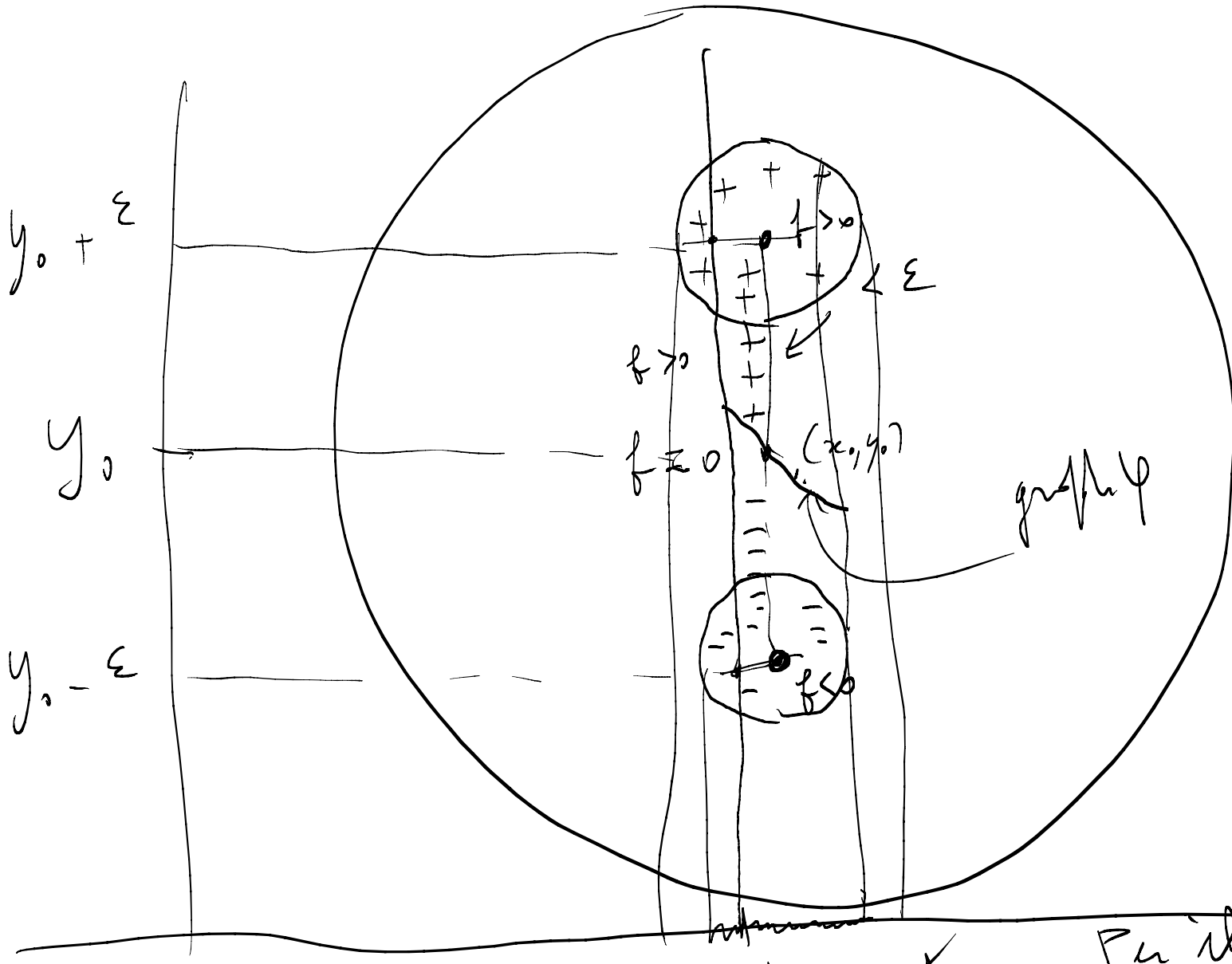
Se x_0 è critico il livello $f(x_0)$ è detto critico

$\varphi(\bar{x})$ = l'unico zero di $y \to f(\bar{x}, y)$

$$f(\bar{x}, \varphi(\bar{x})) = 0$$

$\forall \varepsilon > 0 \; \exists \delta : \; |x - x_0| < \delta \quad |\varphi(x) - \varphi(x_0)| < \varepsilon$





$f(\bar{x}, y_0 + \varepsilon) > 0$

$f(\bar{x}, y_0 - \varepsilon) < 0$

il segmento

$\bar{x}, y \quad , \quad y \in [y_0 - \varepsilon, y_0 + \varepsilon]$

$\subseteq \Omega$ perché

il cerchio è contenuto in Ω

e contenuto in Ω

Per il th. dei

$\bar{x}$ dipendente $\leftarrow \varphi(\bar{x}) \in]y_0 - \varepsilon, y_0 + \varepsilon[$

Per continuare $|\varphi(x) - y_0| < \varepsilon$ $\forall x : |x - x_0| < \delta$

$y_0 = \varphi(x_0)$

numero dei due raggi

$A : X \to Y$

$e_1 \dots e_n$ base in X

$e'_1 \dots e'_m$ base in Y

$A(e_i) = A_{ji}\, e'_j \equiv$

$\equiv \sum_{j=1}^{m} A_{ji}\, e'_j$

AL_6.4

+ AL_6.2 esempi

$A : X \to Y$ $\quad \dim X < +\infty$

$X = \langle u_1 \dots u_n \rangle$

$A(X) = \langle A(u_1), A(u_2), \dots, A(u_n) \rangle$ $\in$

$\forall x \in X \quad \exists \alpha_i : \quad x = \sum_1^n \alpha_i \cdot u_i \Rightarrow A(x) = \sum_1^n \alpha_i A(u_i)$

combinazioni lineari delle immagini di $u_1 \dots u_n$.

$\Rightarrow A(X) \subseteq \langle A(u_1) \dots A(u_n) \rangle$ $\quad$ TESI

Se anche $A(u_1), A(u_2) \dots, A(u_n) \in A(X) \Rightarrow \langle A(u_1) \dots A(u_n) \rangle \subseteq A(X)$

Applicazioni lineari su spazi di dimensione finita

$f(x) = x^2 \qquad f'(x) = 2x$

$$\Lambda(\text{graph } f) = \int_0^1 \sqrt{1 + 4x^2}\, dx \overset{2x=t}{=\!=\!=} \int_0^2 \sqrt{1+t^2}\, \frac{1}{2}\, dt =$$

$x = \frac{t}{2}$

$dx = \frac{1}{2} dt$

$$= \frac{1}{2} \int_0^2 \sqrt{1+t^2}\, dt \overset{t=\sinh u}{=\!=\!=} \frac{1}{2} \int_0^{\sinh^{-1} 2} |\cosh u| \cosh u\, du \overset{\cosh u > 0}{=\!=\!=}$$

$dt = \cosh u\, du$

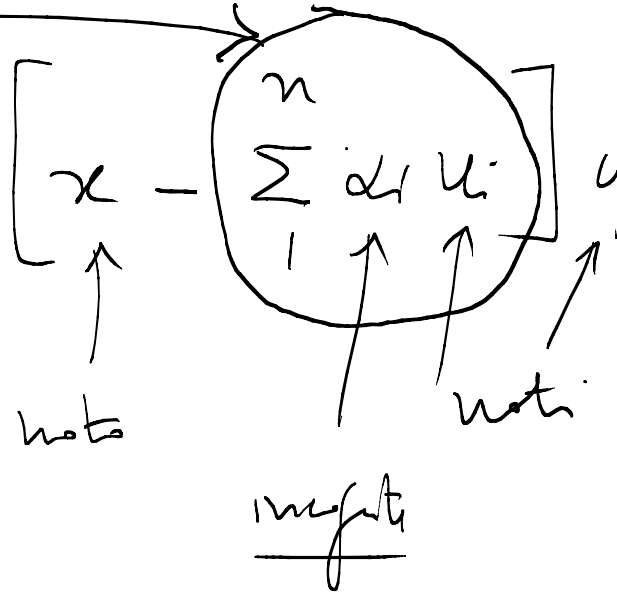
$1 + t^2 = \cosh^2 u$

$x^2 - y^2 = 1$

$$= \frac{1}{2}\int_0^{\sinh^{-1}2} \left[\frac{1}{2}\left(e^u + e^{-u}\right)\right]^2 du = \frac{1}{8}\int_0^{\sinh^{-1}2} \left(e^{2u} + e^{-2u} + 2\right) du = \dots$$

funzioni iperboliche
dispense tra quelle
di Analisi I

$x_{\langle u_1 \dots u_n \rangle} = \sum_1^n \alpha_i u_i$ con $\alpha_1 - \alpha_n$ soluzione qualunque del sistema



$\forall j = 1 \dots n$

→ è risolubile, per il th. di Gram se $u_1 - u_n$ indip.

Per l'unicità + th. Gram se sono ortogonali

$f(x,y) = xy$ $\quad \nabla f = \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow x = y = 0$

L'unico punto stat. è $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$; il hello vitro $\neq 0$

$$Hf(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

indefinito.

det. hessiano $= -1$

$\| \quad \lambda_1 \lambda_2 \Rightarrow \lambda_1 e \lambda_2$ sono discordi!

SELLA NON DEGENERE

$$\underbrace{|x_0 + tv - x_0|}_{\|} < \delta$$

$$|t||v| < \delta$$

$$|t| < \frac{\delta}{|v|}$$