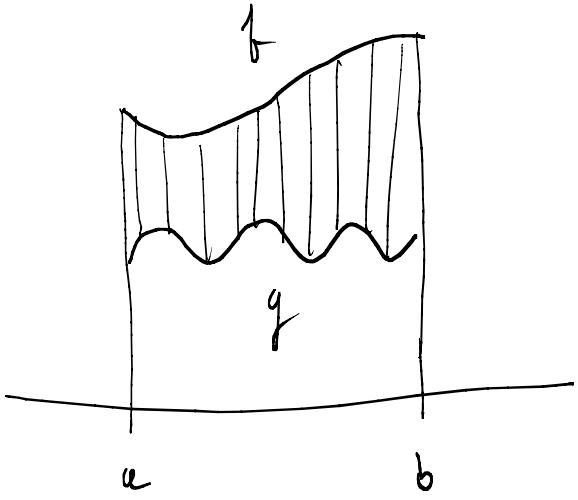
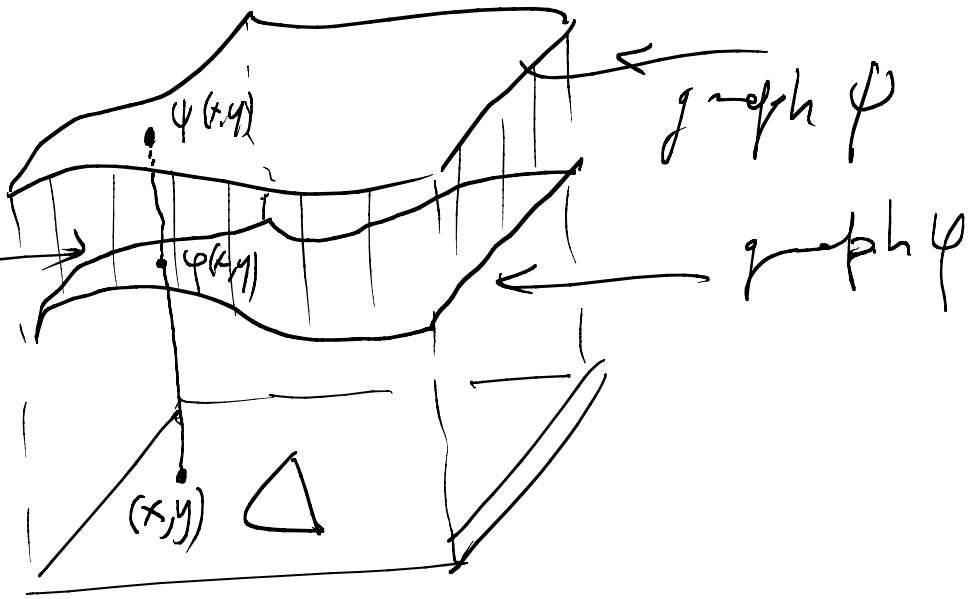
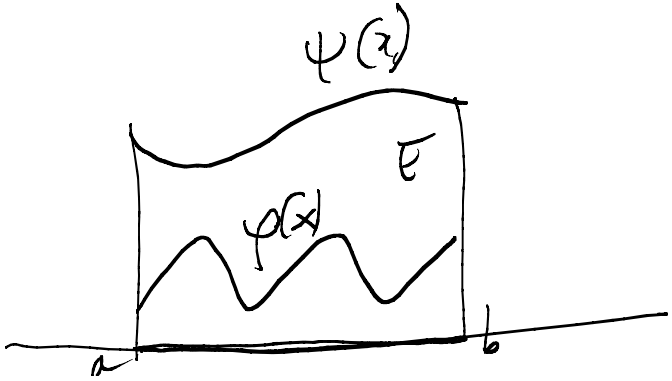
Note Title 6/1/2021



$$\int_a^b (f-g)\,dx$$



$$\text{Vol} = \int_E 1 = |E|$$



$E = \{(x,y) : x \in [a,b],\ y \in [\varphi(x), \psi(x)]\}$

domain with respect to x

$$E = \{(x,y,z) : (x,y) \in \Delta : z \in [\varphi(x,y), \psi(x,y)]\}$$

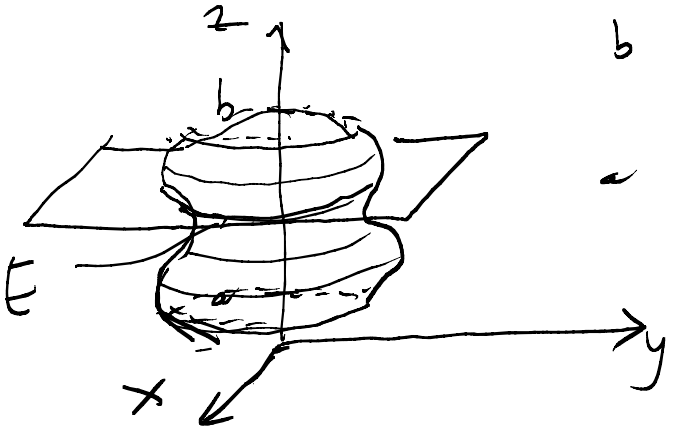
$$\int_E 1 = \int dx\,dy \int_{\Pi(E)_{(x,y)} (\equiv \Delta)}^{E(x,y)} 1\,dz = \int_\Delta \psi(x,y) - \varphi(x,y)\,dx\,dy$$

$E(x,y) : [\varphi(x,y), \psi(x,y)]$

Integrazione per "fili"

VOLUME COMPRESO FRA GRAFICI

VOLUME DI UN SOLIDO DI ROTAZIONE.



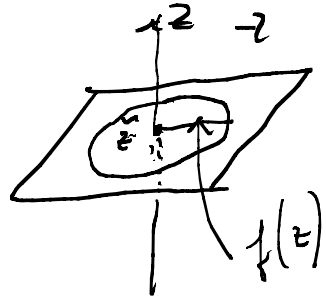
b

a

$y = f(z)$ $\quad f \geq 0$

proiettando E sull'asse z: $\Pi_z E = [a,b]$

$$E(\bar{z}) = \{(x,y,\bar{z}) : x^2 + y^2 \leq f^2(\bar{z})\}$$



$$|E| = \int_a^b dz \int 1 \, dx dy = \int_a^b \pi f^2(z) \, dz$$

$E(z)$

$\pi f^2(z)$

per ultimo l'integrale 1 dimensionale (dz)

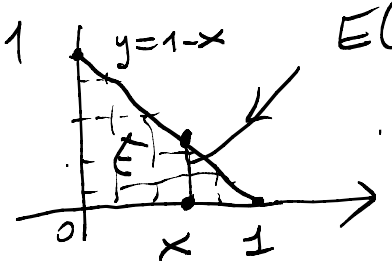
area del cerchio "generico" di raggio $f(z)$

per primo l'integrale sulla "fetta" $E(z)$

INTEGRAZIONE PER "STRATI"

$$\int x^2 e^{xy} \, dx dy = \int_0^1 dx \int_0^{1-x} x^2 e^{xy} \, dy =$$

$[0,1] = \pi_x E$



$$= \int_0^1 x^2 dx \int_0^{1-x} e^{xy} dy = \int_0^1 x^2 dx \left[\frac{1}{x} e^{xy}\right]_0^{1-x} =$$

x costante in y

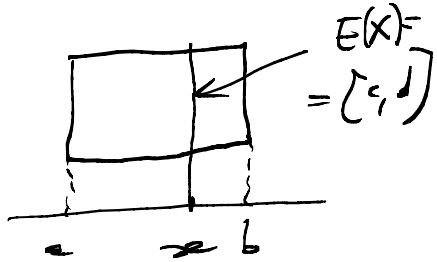
$$\int e^{xy} dy = \frac{1}{x} e^{xy}$$

$$\left(\frac{1}{x} e^{xy}\right)_y = e^{xy}$$

$$= \int_0^1 x^2 \frac{1}{x} \left(e^{x(1-x)} - 1\right) dx = \int_0^1 \underbrace{x\, e^{x-x^2}}_{} - x\, dx = \dots$$

$\rightarrow f(x,y) = g(x)\, h(y)$ funzioni "ideali" per l'integrale

$\rightarrow E = [a,b] \times [c,d]$

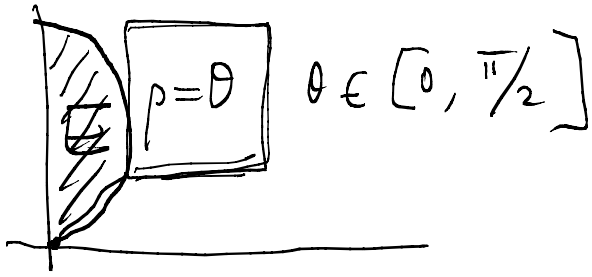


$$\int_E f(x,y)\, dy = \int_a^b dx \int_c^d dy\, g(x)h(y) = \boxed{\int_a^b dx\, g(x)} \boxed{\int_c^d dy\, h(y)} \leftarrow$$

g è costante in y

Coordinate polari (piane)

$$\begin{cases} x = \rho \cos\theta \\ y = \rho \sin\theta \end{cases} \qquad |\det| = \rho$$



Porzione di sfera di Archimede relativa a $\theta \in [0, \pi/2]$

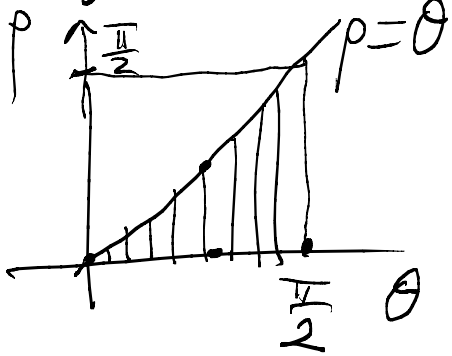
$$|E| = \int 1 \, dx\,dy =$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\theta} \rho \, d\rho \overset{(\rho=\theta)}{=} \int_0^{\frac{\pi}{2}} \frac{1}{2}\theta^2 \, d\theta$$

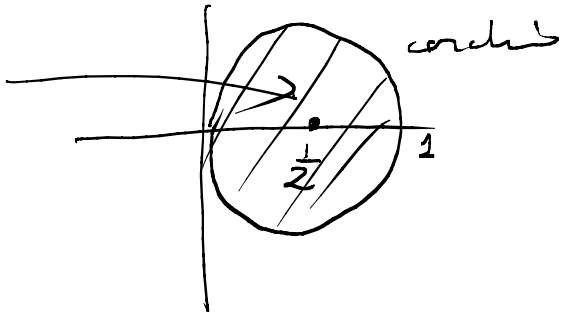
$|\text{jacobiano}|$ della trasformazione $(\rho, \theta) \to (x, y)$

Sulla sfera unità

$$x(\theta) = \theta\cos\theta$$
$$y(\theta) = \theta \sin\theta$$



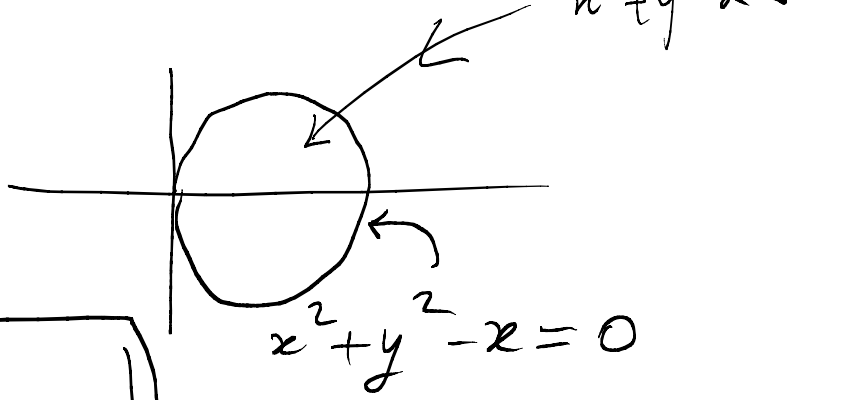
$$\int_E \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy$$



$x = \rho\cos\theta$

$y = \rho\sin\theta$

$$\frac{1}{\sqrt{x^2+y^2}} \overset{\rho>0}{=} \frac{1}{\rho}$$



$$E = \{(x,y)\in\mathbb{R} : x^2+y^2-x<0\}$$

$$x^2+y^2-x=0$$

$$\left(x-\frac{1}{2}\right)^2+y^2-\frac{1}{4}=0$$

$$\left(x-\frac{1}{2}\right)^2+y^2=\frac{1}{4}$$

$$\rho^2\cos^2\theta+\rho^2\sin^2\theta-\rho\cos\theta<0$$

$$\rho^2<\rho\cos\theta \quad \text{div. per } \rho>0$$

$d((x,y),(\frac{1}{2},0))$

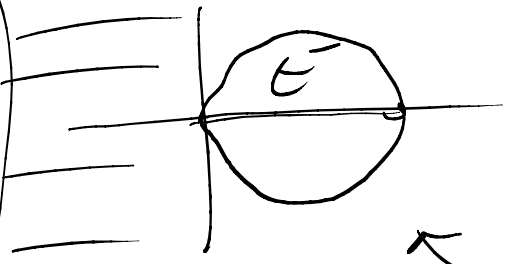
$\left(\frac{1}{2}\right)^2$

$$\rho<\cos\theta$$

$\Longrightarrow$ $\cos\theta > 0$ perché $\cos\theta > \rho \geq 0$

$\Updownarrow$

$\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$



$\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

$0 < \rho < \cos\theta$

Il dominio in coord. polari è normale rispetto a θ d'ordine 1.

$\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ $\quad \rho \in \underset{\uparrow \varphi(\theta)}{0}, \underset{\searrow \psi(\theta)}{\cos\theta}\left[\right.$

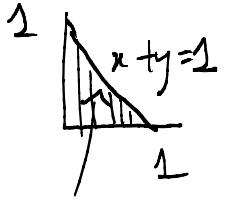
$$\int_E \frac{1}{\sqrt{x^2+y^2}}\, dx\,dy =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{\cos\theta} d\rho\, \frac{1}{\rho}\, \rho \quad \leftarrow \text{jacobiano del cambio di variabile}$$

$\downarrow$ funzione integranda

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\, (\cos\theta - 0) = \dots$$

$$\int \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy$$



E

$$E=\{(x,y): x>0;\ y>0;\ x+y<1\}$$

passando a coord. polari

$$\begin{cases} \rho\cos\theta>0 \Rightarrow \cos\theta>0 \\ \rho\sin\theta>0 \Rightarrow \sin\theta>0 \\ \rho\cos\theta+\rho\sin\theta<1 \\ \rho(\cos\theta+\sin\theta)<1 \end{cases}$$

$\theta\in\left[0,\frac{\pi}{2}\right]$

somma di ≥ 0 ma diminuisce. multi

$$\rho<\frac{1}{\cos\theta+\sin\theta}$$

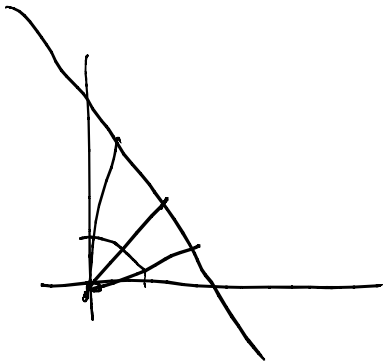
$\rho>0$

$\rho\in\left]0,\ \frac{1}{\cos\theta+\sin\theta}\right[$

$$\int d\theta \int d\rho\ \rho\ \frac{1}{\rho}$$

ρ funzione

$|\text{jacobiano}|$



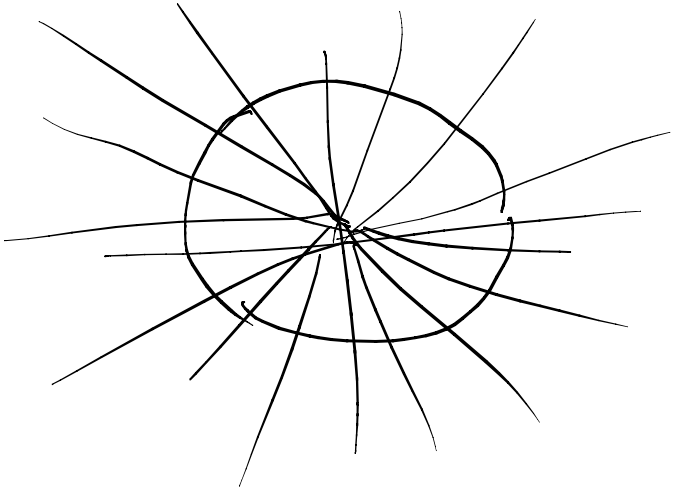
$$\int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{1}{\cos\theta+\sin\theta}} d\rho = \int_0^{\frac{\pi}{2}} \frac{1}{\cos\theta+\sin\theta}\,d\theta =$$

$\tan\frac{\theta}{2}=t$

$\cos\theta=\frac{1-t^2}{1+t^2}$

$\sin\theta=\frac{2t}{1+t^2}$

$$\int_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy = \int_0^{2\pi} d\theta \int_0^{+\infty} e^{-\rho^2}\,\rho\,d\rho$$



$$x^2+y^2=\rho^2$$

$$2\pi \int_0^{+\infty} e^{-\rho^2}\left(-\frac{1}{2}\right)(-2\rho)\,d\rho \underset{-\rho^2=t}{=} -\pi \int_0^{-\infty} e^{-t}\,dt =$$

$$dt=-2\rho\,d\rho$$

$$= -\pi\left(-e^{-t}\Big|_0^{-\infty}\right) = -\pi\,(-1) = \pi$$

$$\int_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy = \pi$$

$$\Big\| \quad \text{Fubini}$$

$$\underbrace{\int_{\mathbb{R}} e^{-x^2}\,dx}_{k} \cdot \underbrace{\int_{\mathbb{R}} e^{-y^2}\,dy}_{k}$$

$$k^2 = \pi \Rightarrow \boxed{\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}}$$

$$e^{-x^2-y^2} = e^{-x^2}\,e^{-y^2}$$

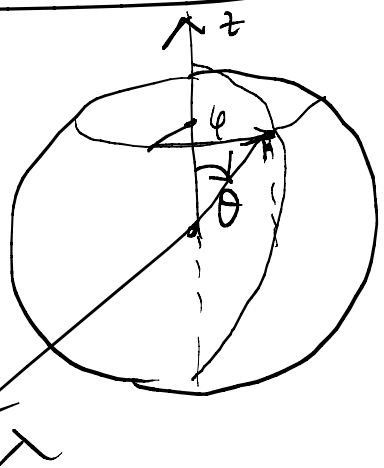
$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$$

COORDINATE SFERICHE

$$\begin{cases} x = \rho \sin\theta \cos\varphi \\ y = \rho \sin\theta \sin\varphi \\ z = \rho \cos\theta \end{cases} \qquad J = \rho^2 |\sin\theta| = \rho^2 \sin\theta$$

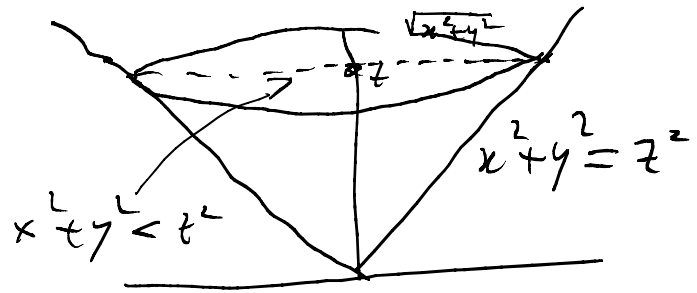
$$\boxed{\theta \in [0,\pi]} \qquad \varphi \in [0, 2\pi]$$

$$\rho = \sqrt{x^2+y^2+z^2}$$



$$\left| \{x^2+y^2 < z^2\} \cap \{x^2+y^2+z^2 < 2z\} \right|$$

$\cap \{z>0\}$ inutile perché segue da



$\rho^2 \sin^2\theta \cos^2\varphi + \rho^2 \sin^2\theta \sin^2\varphi < \rho^2 \cos^2\theta$

$\rho > 0$

$\sin^2\theta < \cos^2\theta$

$\Updownarrow$

$|\sin\theta| < |\cos\theta|$

$\theta \in [0,\pi]$

$\theta \in \left]-\frac{\pi}{4}, \frac{\pi}{4}\right[\cap [0,\pi] = \left[0, \frac{\pi}{4}\right[$

$\theta \in \left]\frac{3}{4}\pi, \frac{5}{4}\pi\right[$

$x^2+y^2+(z-1)^2-1 < 0$

$x^2+y^2+(z-1)^2 < 1$

quadrato della distanza da $(0,0,1)$

$\leq$ quadrato del raggio

sfera di centro $(0,0,1)$ e raggio 1

$x^2+y^2+z^2 < 2z$

$\rho^2 < 2\rho\cos\theta \Rightarrow$ $\rho > 0$

$\boxed{\rho < 2\cos\theta} \Rightarrow \boxed{\cos\theta > 0}$ $\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

θ coordinata sferica $\in [0,\pi]$

$\theta \in \left]0, \frac{\pi}{4}\right[\quad \varphi \in [0, 2\pi] \quad 0 < \rho < 2\cos\theta$

(↓ parametri; ultimo: retta.)

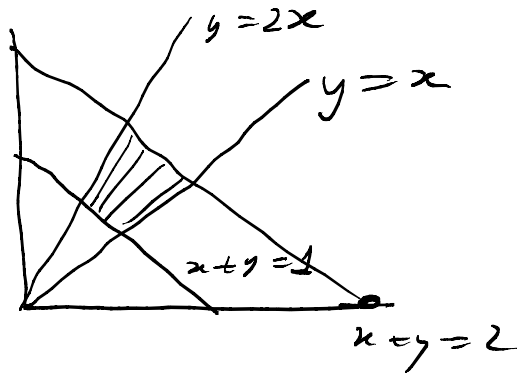
$$\int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{4}} d\theta \int_0^{2\cos\theta} \underbrace{\rho^2 \sin\theta}_{\text{Jacobiano coord. sferiche}} d\rho = 2\pi \int_0^{\frac{\pi}{4}} d\theta \sin\theta \int_0^{2\cos\theta} \rho^2 d\rho =$$

($\int_0^{2\pi} d\varphi$)

$$= 2\pi \int_0^{\frac{\pi}{4}} d\theta \sin\theta \, \frac{1}{3}\rho^3 \Big|_0^{2\cos\theta} = 2\pi \int_0^{\frac{\pi}{4}} d\theta \sin\theta \, \frac{1}{3} 8\cos^3\theta =$$

$$\int_E \frac{dx\,dy}{xy}$$

E



$$1 < \underbrace{x+y}_{u} < 2 \qquad 2 > \underbrace{\frac{y}{x}}_{v} > 1$$

$$\begin{cases} u = x+y \\ v = \dfrac{y}{x} \end{cases}$$

$-x = y-u \Rightarrow x = u-y$ $\qquad y = u-x$

$v = \dfrac{y}{x}$ $\qquad y = vx$

$u - x = vx \Rightarrow (1+v)x = u$

$y = vx = \dfrac{uv}{1+v}$ $\qquad x = \dfrac{u}{1+v}$

$$\frac{\partial x}{\partial u} = \frac{1}{1+v} \qquad \frac{\partial x}{\partial v} = -\frac{u}{(1+v)^2}$$

$$\frac{\partial y}{\partial u} = \frac{v}{1+v} \qquad \frac{\partial y}{\partial v} = \frac{u(1+v) - uv}{(1+v)^2} = \frac{u}{(1+v)^2}$$

$$\det j = \begin{vmatrix} \frac{1}{1+v} & -\frac{u}{(1+v)^2} \\ \frac{v}{1+v} & \frac{u}{(1+v)^2} \end{vmatrix} = \frac{u}{(1+v)^3} + \frac{uv}{(1+v)^3} =$$

$$= \frac{u(1+v)}{(1+v)^3} = \frac{u}{(1+v)^2} = |\det j| \quad \begin{matrix} u \in [1,2] \\ v \in [1,2] \end{matrix}$$

$$\int_1^2 du \int_1^2 dv \; \frac{1}{\frac{u^2 v}{(1+v)^2}} \cdot \frac{u}{(1+v)^2} = \int_1^2 du \int_1^2 dv \; \frac{1}{uv} = \int_1^2 du \, \frac{1}{u} \int_1^2 dv \, \frac{1}{v} =$$

$$= \log 2 \cdot \log 2 = (\log 2)^2$$

forma integrale

Formula di Gauss-Green - Ostrogradskij

$$\int_\Omega f_x \, dx dy = \int_{\partial\Omega} f \, dy \qquad \int_\Omega f_y \, dx dy = -\int_{\partial\Omega} f \, dx$$

Ω integrale doppio

$\partial\Omega$ integrale di forma diff.

$$|E| = \int 1 \, dx dy = \int x \, dy$$

Green ∂E

integrale di forma diff.

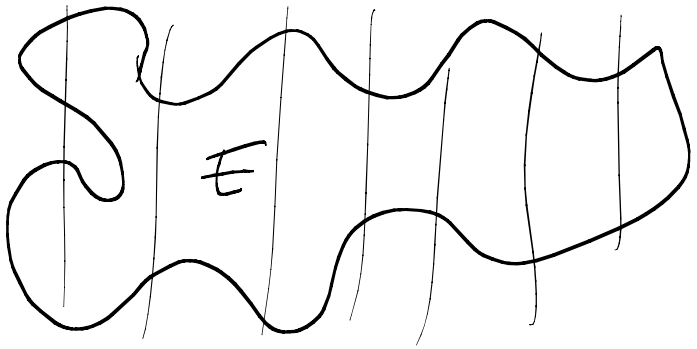
$$1 = \frac{\partial}{\partial x} x$$

$$f = x$$

$$1 = \frac{\partial f}{\partial x}$$

$$g = y$$

$$\frac{\partial g}{\partial y} = 1$$



Due metodi diversi di calcolare la misura.

$$|E| = -\int y \, dx$$

$$|E| = \frac{1}{2}\int x\,dy - y\,dx$$

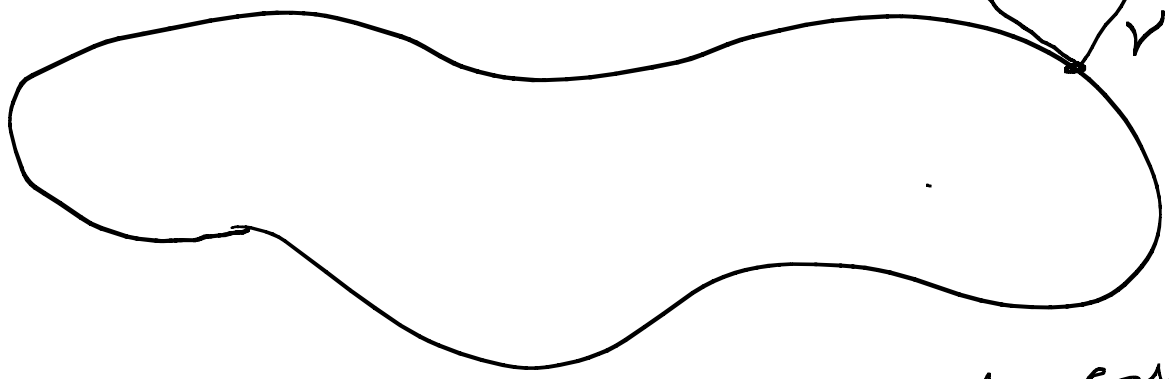
∂E

$\gamma = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

$$\int_{\text{cerchio}} 1 \, dx dy = \int_X x\,dy = \int_0^{2\pi} \cos t \cos t \, dt = \int_0^{2\pi} \cos^2 t \, dt \leftarrow \cos 2t = 2\cos^2 t - 1$$

$$\int_\Omega f_{x_i}\,dx = \int_{\partial\Omega} f\,\nu_i\,dl$$

ν versore normale esterno



ortogonale

$$\gamma(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\dot\gamma(t) = \begin{pmatrix} \dot x(t) \\ \dot y(t) \end{pmatrix}$$

$$\nu = \frac{1}{|\dot\gamma|}\begin{pmatrix} \dot y(t) \\ \dot x(t) \end{pmatrix} = \begin{pmatrix} -\dot y(t) \\ \dot x(t) \end{pmatrix}\frac{1}{|\dot\gamma|}$$

$-y\,dx$ $\quad$ $x\,dy$

Area di una superficie parametrica

$$\phi : \Delta \to \mathbb{R}^3$$

$$\text{Area } \phi = \int_\Delta |\underbrace{\phi_u \times \phi_v}_{\text{vettore normale standard}}| \, du\,dv$$

$$\Lambda = \int |\gamma'(t)| \, dt$$

caso cartesiano.

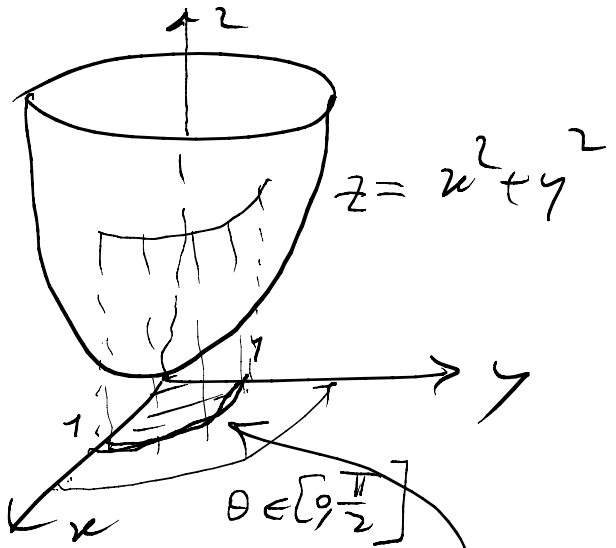
$$\Lambda(\text{graph } f) = \int \sqrt{1+f'^2}\, dt$$

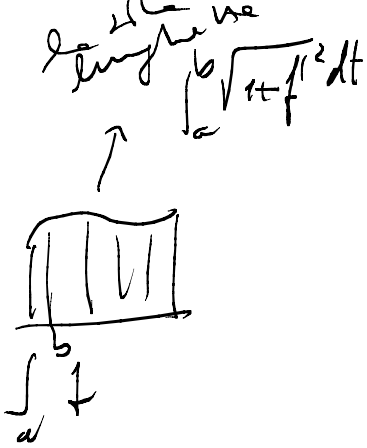
ϕ superficie cartesiana $\quad u, v \in \Delta$

$$\begin{pmatrix} u \\ v \\ f(u,v) \end{pmatrix} \to \begin{pmatrix} x = u \\ y = v \\ z = f(u,v) \end{pmatrix}$$

$$\text{Area}(\text{sup. cart.}) = \int_\Delta \sqrt{1 + f_u^2 + f_v^2}$$

$f(x,y) = x^2 + y^2$





$f_x = 2x$

$f_y = 2y$

$$\iint \sqrt{1 + 4x^2 + 4y^2}\,dx\,dy = \iint \sqrt{1 + f_x^2 + f_y^2}\,dx\,dy$$

$$\begin{cases} x^2 + y^2 \le 1 \\ x > 0 \\ y > 0 \end{cases}$$

$x = \rho\cos\theta$

$y = \rho\sin\theta$

$J = \rho$

$x^2 + y^2 = \rho^2$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^1 d\rho\, \sqrt{1+4\rho^2}\,\rho \quad \begin{matrix} dt = 8\rho\,d\rho \\ = \\ t = 1 + 4\rho^2 \end{matrix} \quad \cdots\cdots$$