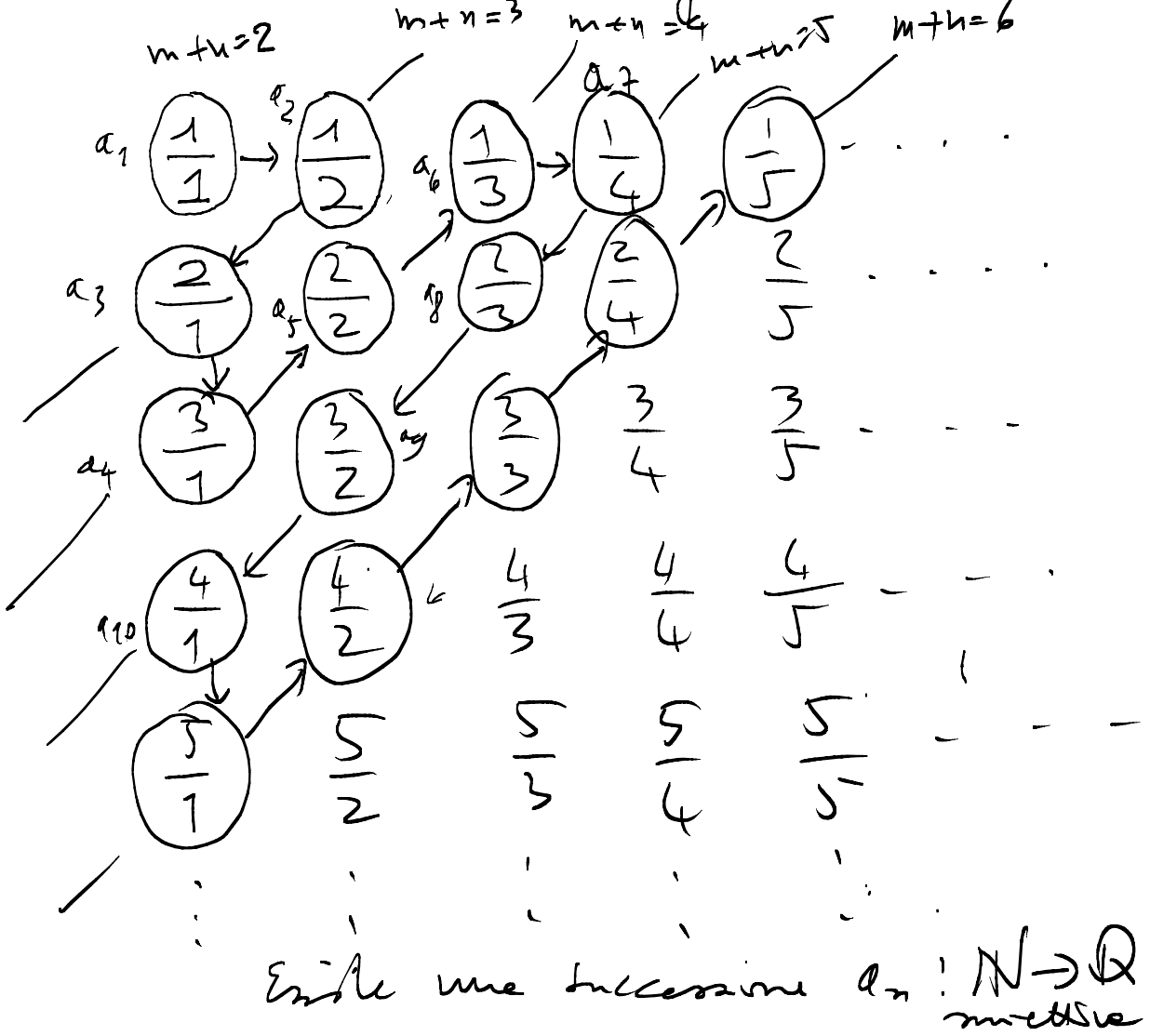
Note Title 5/26/2021

Procedimento diagonale di Cantor



Esiste una successione $a_n : \mathbb{N} \to \mathbb{Q}$ suriettiva

$$\frac{m}{n} = \frac{2m}{2n} = \frac{3m}{3n} \dots$$

Contiene TUTTE LE FRAZIONI A TERMINI POSITIVI

Le frazioni $\frac{m}{n}$ per cui $m+n = k$ fisso sono un numero finito.

INTEGRABILITÀ

1) $f : E \to \mathbb{R}$ E misur. $m(E) < +\infty$

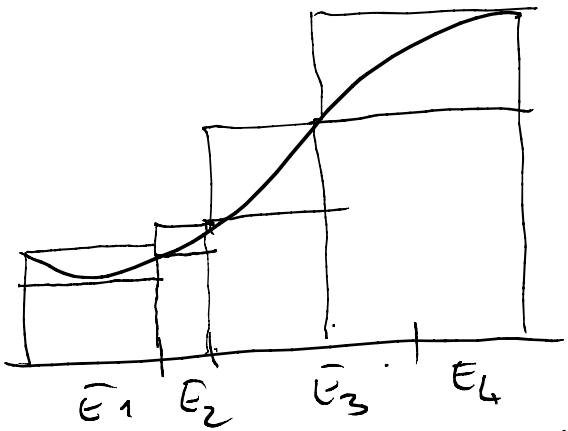
f LIMITATA

$\Pi = \{E_1, E_2, \dots, E_N\}$ partizione cioè $\begin{cases} \bigcup_1^N E_i = E \\ E_i \cap E_j = \emptyset \end{cases}$

$E_1 \dots E_N$ misurabili.

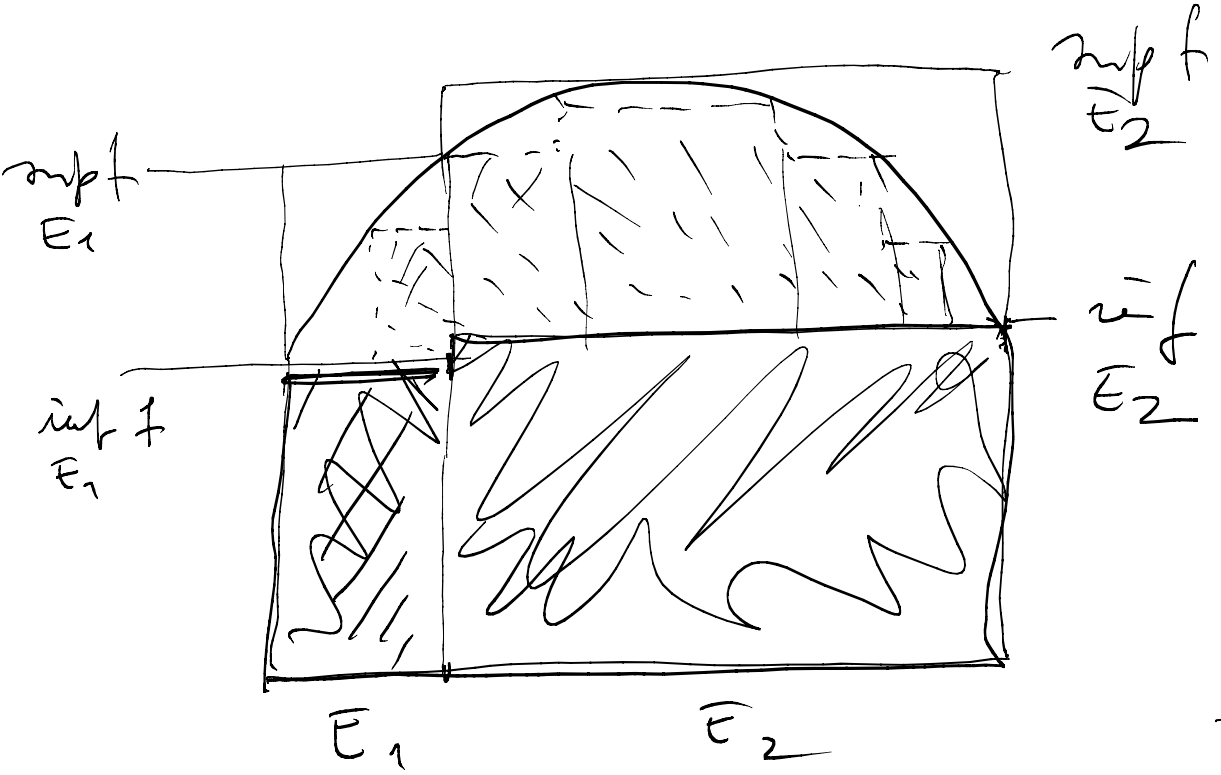
$$\Sigma_*(f) = \sum_{i=1}^{N} m(E_i) \inf_{E_i} f$$

$$\Sigma^*(f) = \sum_{1}^{N} m(E_i) \sup_{E_i} f$$

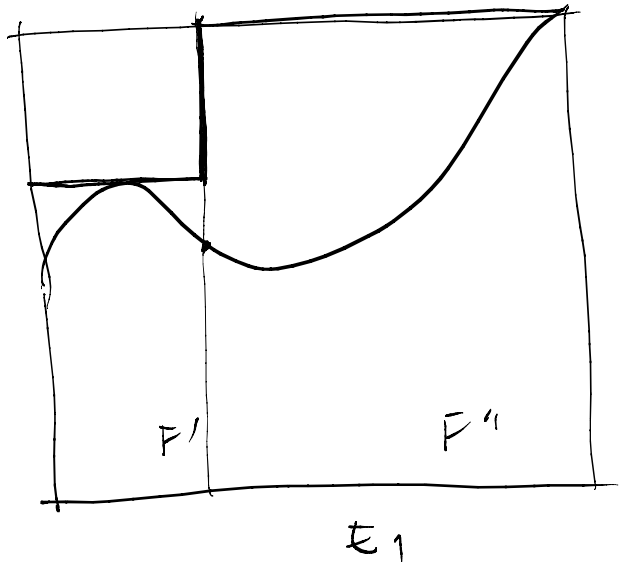


$$\int_*^* f = \sup_{\Pi} \Sigma_* \qquad \int^* f = \inf_{\Pi} \Sigma^*$$

f si dirà integrabile (secondo Lebesgue) su E se $\int_*^* f = \int^* f$ e il valore comune si denoterà con $\int_E f(x)dx$ ed è L'INTEGRALE



Raffinando la partizione, la somma inferiore CRESCE



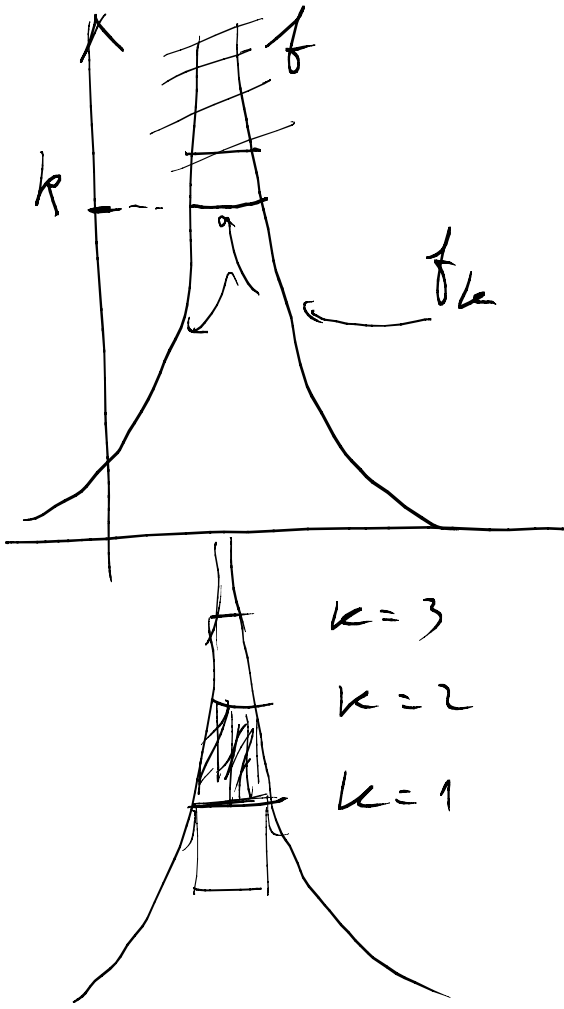
$$\Sigma^*(F_1, F_2) < \Sigma^*(E_1)$$

II Passo

$f : E \to \mathbb{R}$ $\quad \underline{f \geq 0}$ (anche se non limitata)

$|E| < +\infty$

$$f_k(x) = \begin{cases} f(x) & \text{se } f(x) \leq k \\ k & \text{se } f(x) > k \end{cases}$$



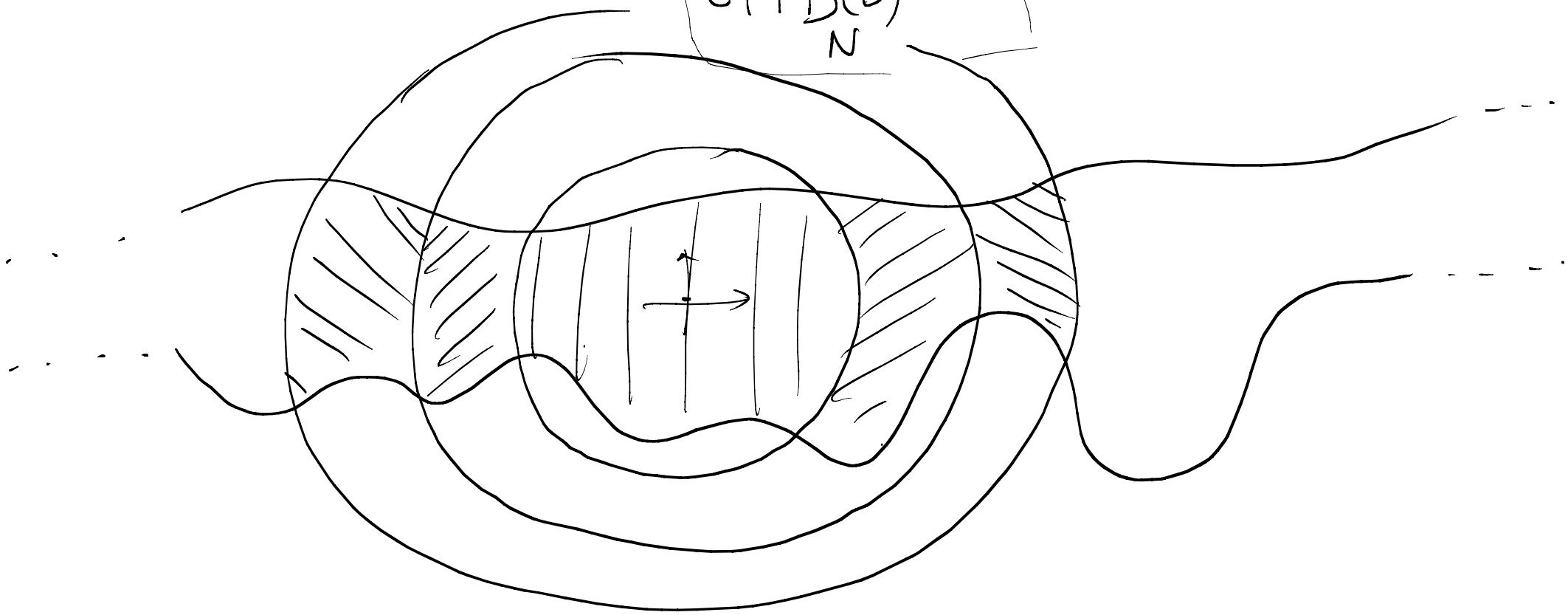
$$\int_E f(x)\,dx = \sup_{k \in \mathbb{N}} \int_E f_k(x)\,dx$$

$\int_E f_k(x)\,dx$: integrale di una funzione limitata, definito prima.

$\int f \leq \int g$ se $f \leq g$ $\qquad \int_E h > 0$ se $h > 0$

III passo $f \geq 0$ su $|E| = +\infty$.

$$\int_E f(x)\,dx = \sup_{N \in \mathbb{N}} \int_{E \cap B(0)_N} f(x)\,dx$$

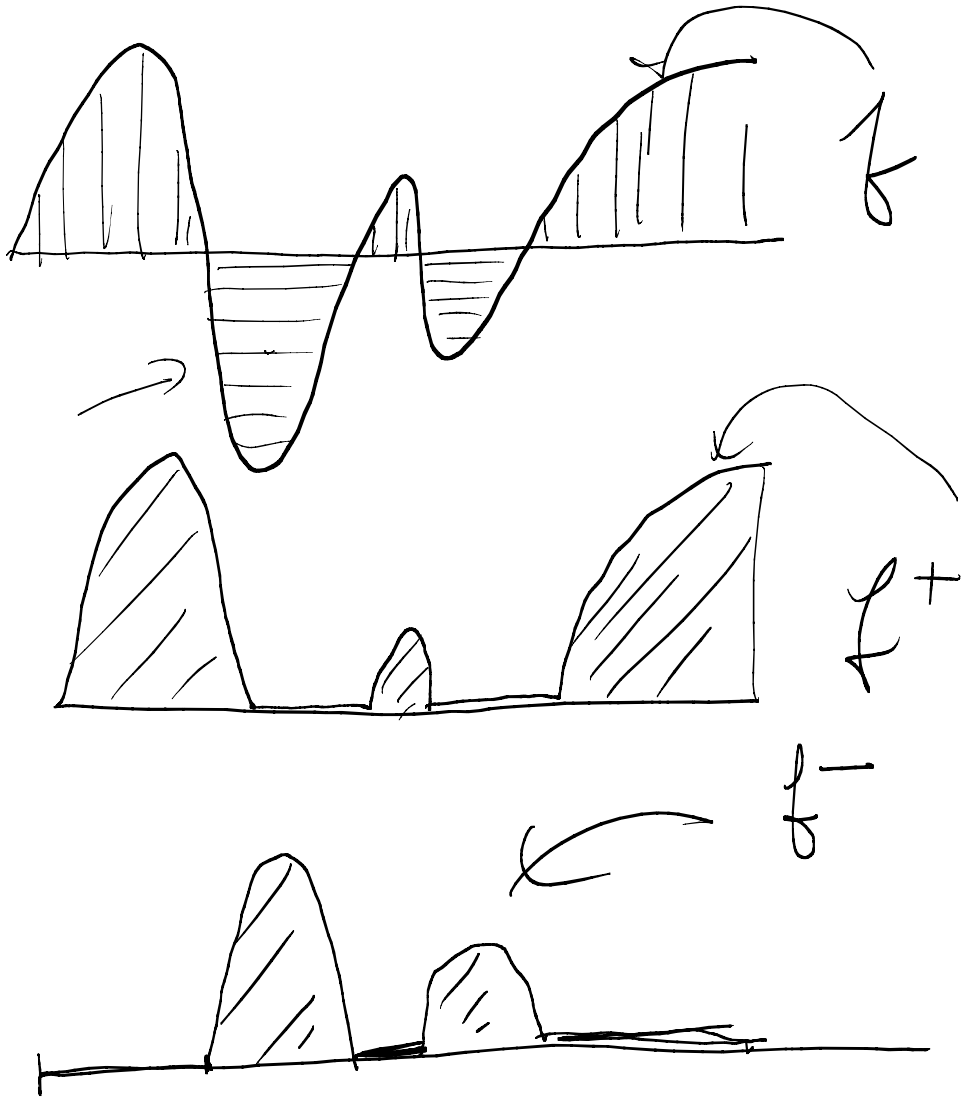


f di segno variabile

$$f^+(x) = \begin{cases} f(x) & f(x) \geq 0 \\ 0 & f(x) < 0 \end{cases}$$

$$f^-(x) = \begin{cases} 0 & f(x) > 0 \\ -f(x) & f(x) \leq 0 \end{cases}$$

$$\boxed{f(x) = f^+(x) - f^-(x)}$$



f^+ ed f^- SONO ENTRAMBE NON NEGATIVE.

Infatti, $|f| = f^+ + f^-$ ← le somme integrali di f^+ ed f^- lo è anche $|f|$

DEF.

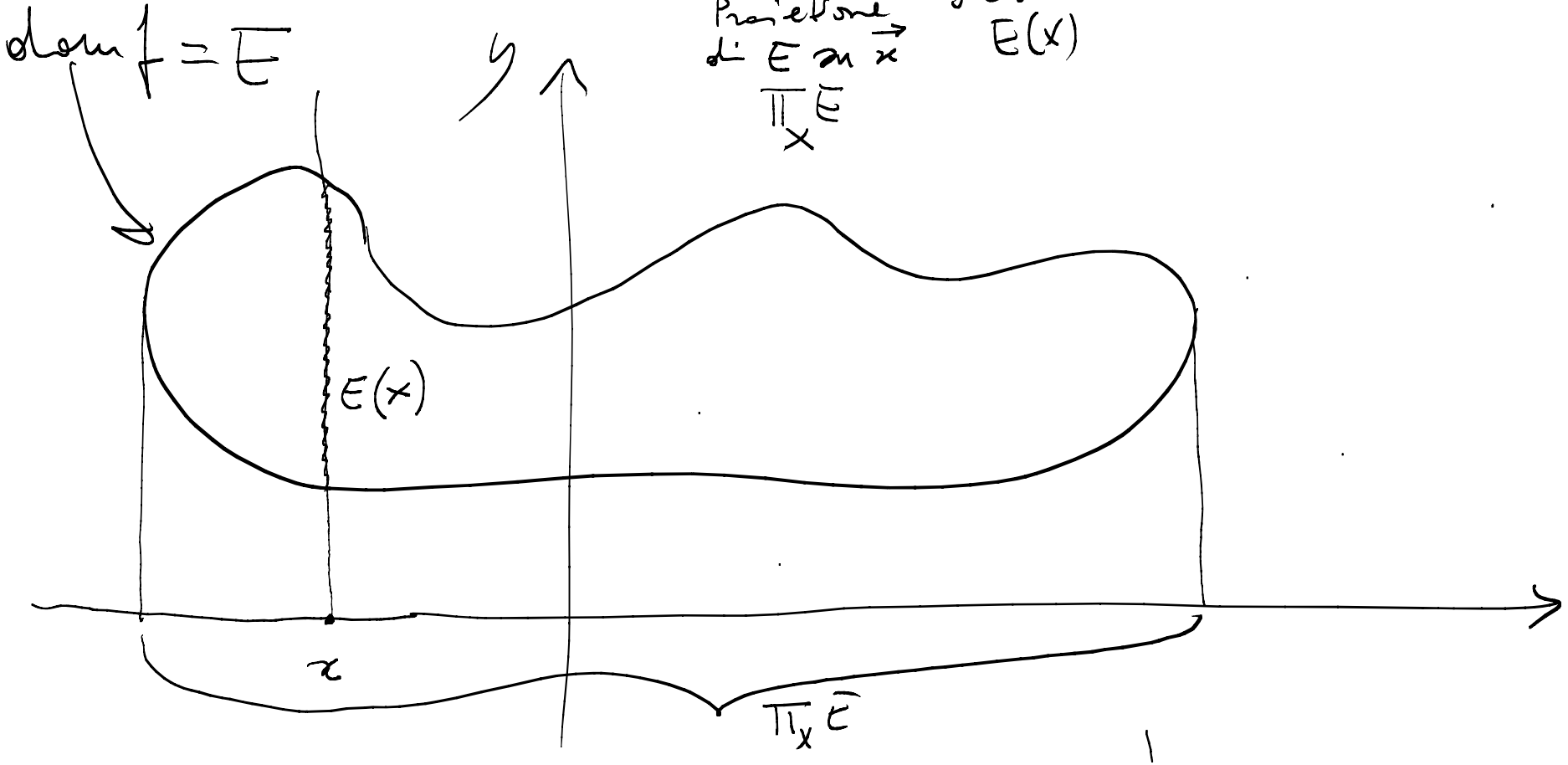
$$\int_E f \equiv \int_E f^+ - \int_E f^-$$

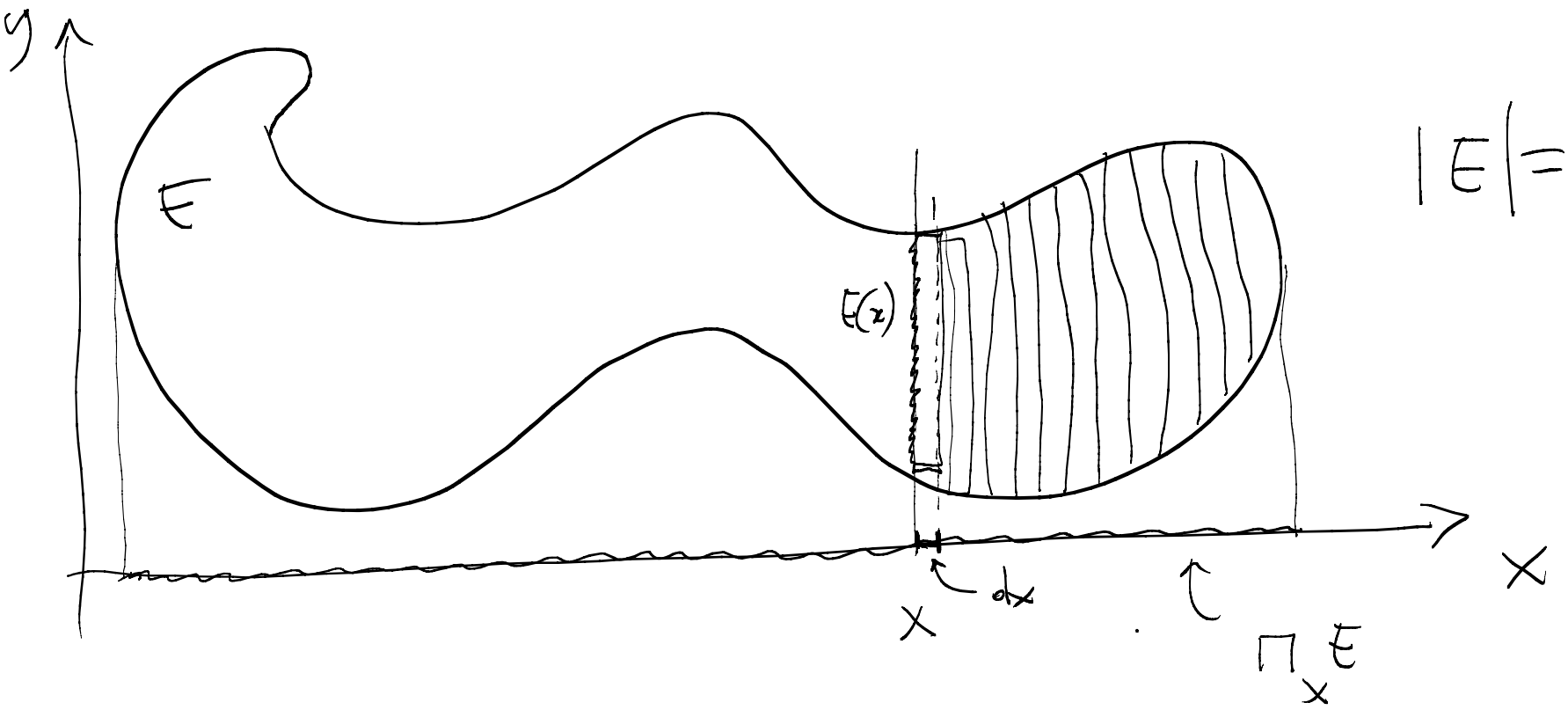
DEF. f sarà detta INTEGRABILE SU E se $\int_E f^+$ è finito, $\int_E f^-$ è finito e $\int_E f = \int_E f^+ - \int_E f^-$

Th. FUBINI / Th. TONELLI

$$\int_E f(x,y)\,dx\,dy = \int_{\text{Proiezione di } E \text{ su } x = \pi_x E} dx \int_{\text{Sezione } E(x)} f(x,y)\,dy \equiv \int_{\pi_x E} \underbrace{\left(\int f(x,y)\,dy\right)}_{\text{funzione di } x} dx$$

x "costante"





$$|E| = \int_E 1 \, dx dy \dots$$

$$\int_{E(x)} 1 \, dx = |E(x)|$$

$$\int_{E(x)} 1 \, dy$$

ANALOGAMENTE $\int_E f \, dx dy = \int_{\Pi_y(E)} dy \int f(x,y) \, dx$

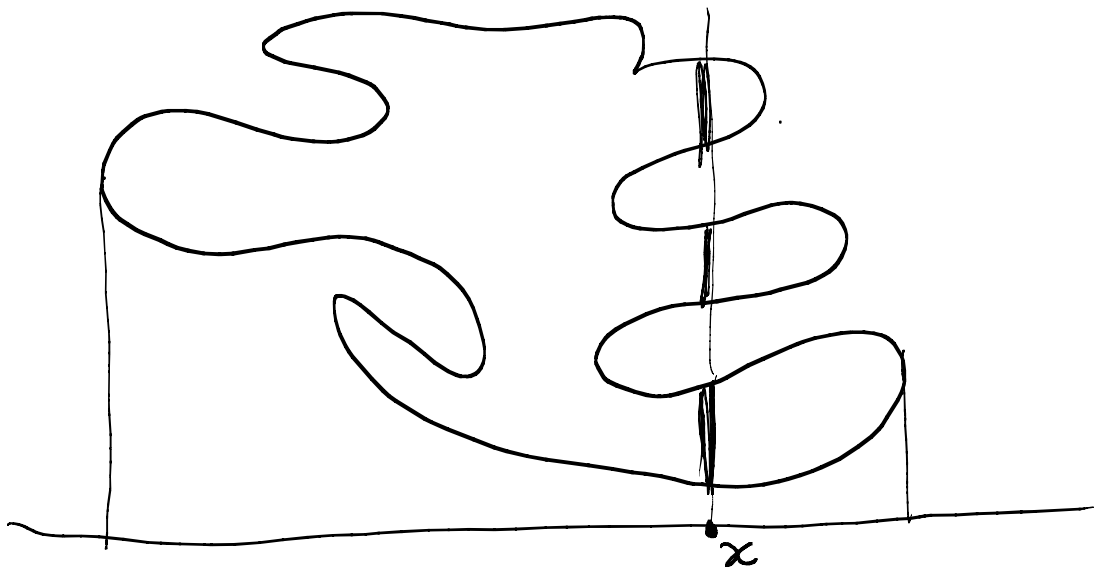
Si può calcolare $\int_E f$ con integrali iterati se

1) (FUBINI) f è integrabile su E

$\left(\int |f| < +\infty\right)$

o ALTERNATIVAMENTE

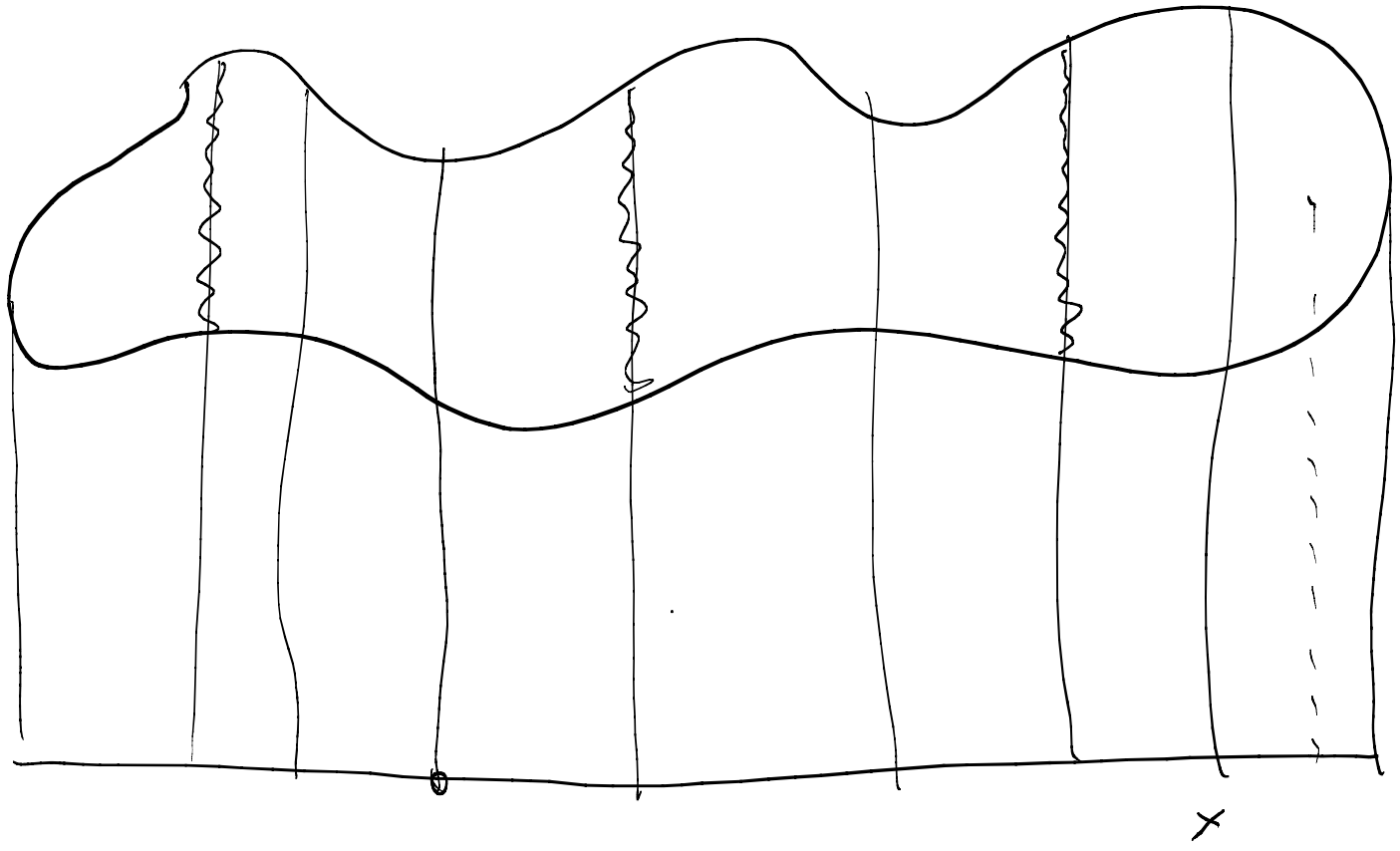
2) (TONELLI) $f \geq 0$



$$E(\bar{x}) = \left\{(x,y) \in E : x = \bar{x}\right\} =$$

$$= E \cap \left\{x = \bar{x}\right\}$$

E sarà detto dominio normale rispetto a x se $E(x)$ è sempre un intervallo $\forall x \in \pi_x(E)$

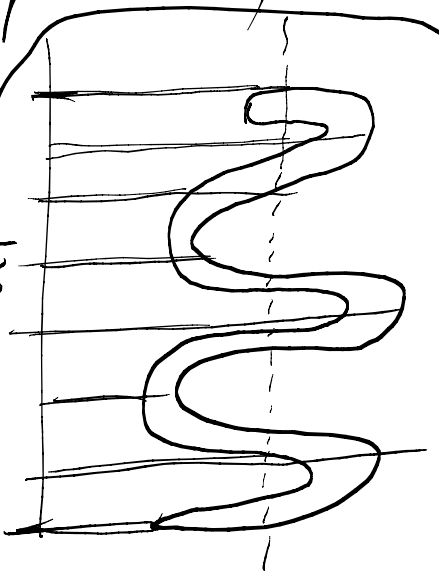


Dato $E \subseteq \mathbb{R}^2$

$$\pi_x E = \{ x \in \mathbb{R} : \exists y \in \mathbb{R} \quad (x,y) \in E \}$$

$$E(x) = \{ y \in \mathbb{R} : (x,y) \in E \}$$

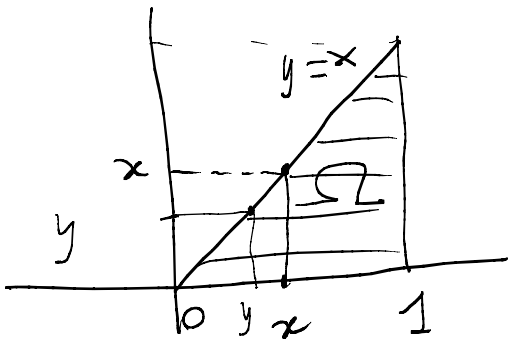
E è detto normale (rispetto all'asse x) se $E(x)$ è un intervallo $(\varphi(x), \psi(x))$ — sempre ($\forall x \in \pi_x E$)



NORMALE RISPETTO A y

DUNQUE E normale ha l'aspetto

$$\{ x \in [a,b] \quad \varphi(x) \le y \le \psi(x) \}$$



$$\int_\Omega xy \, dx\,dy =$$

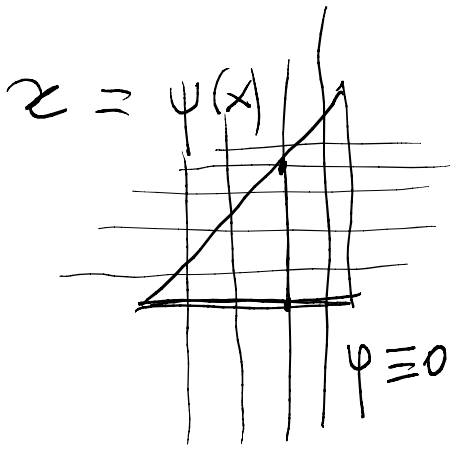
$$= \int_0^1 dx \int_{\varphi(x)\equiv 0}^{\psi(x)=x} xy \, dy =$$

$$= \int_0^1 dx \; x \int_0^x y \, dy =$$

costante in y

$$= \int_0^1 dx \; x \left(\frac{1}{2} y^2 \Big|_0^x \right) = \int_0^1 x \frac{1}{2} x^2 dx = \frac{1}{2} \int_0^1 x^3 dx =$$

$$= \frac{1}{8}$$



Ω è normale anche rispetto ad y

$$\Omega = \{ y \in [0,1] \; ; \; y \le x \le 1 \}$$

$$\int_0^1 dy \int_y^1 xy \, dx$$

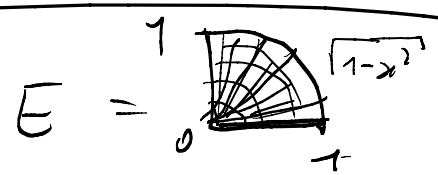
$g: \Sigma \to \Omega \qquad \Sigma, \Omega \subseteq \mathbb{R}^n$

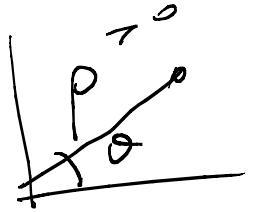
$g \in C^1$ invertibile con inversa C^1

$$\int_\Omega f(y)\,dy = \int_{y=g(x)} f(g(x))\,\left|\det g'(x)\right|\,dx$$

$\Sigma = g^{-1}(\Omega)$

$$\int_E \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy =$$

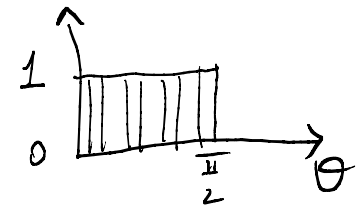




$x = \rho\cos\theta$

$y = \rho\sin\theta$

det jacobiano $= \rho$

$$= \int_0^{\pi/2} d\theta \int_0^1 d\rho\; \underbrace{\frac{1}{\rho}}_{f(g(\rho,\theta))}\;\underbrace{\rho}_{|\det g'|} = \int_0^{\frac{\pi}{2}} d\theta \int_0^1 d\rho = \boxed{\frac{\pi}{2}}$$

$\sqrt{x^2+y^2} = \sqrt{\rho^2\cos^2\theta + \rho^2\sin^2\theta} = \sqrt{\rho^2} = \rho$