Note Title 5/25/2021

MISURA DI UN SOTTINSIEME DI $\mathbb{R}^n$

(lunghezza in $\mathbb{R}^1$, area in $\mathbb{R}^2$, volume in $\mathbb{R}^n$, $n \geq 3$)

$$3 + \frac{10}{71} < \pi < 3 + \frac{1}{7}$$

LEBESGUE (pron. lebèg)

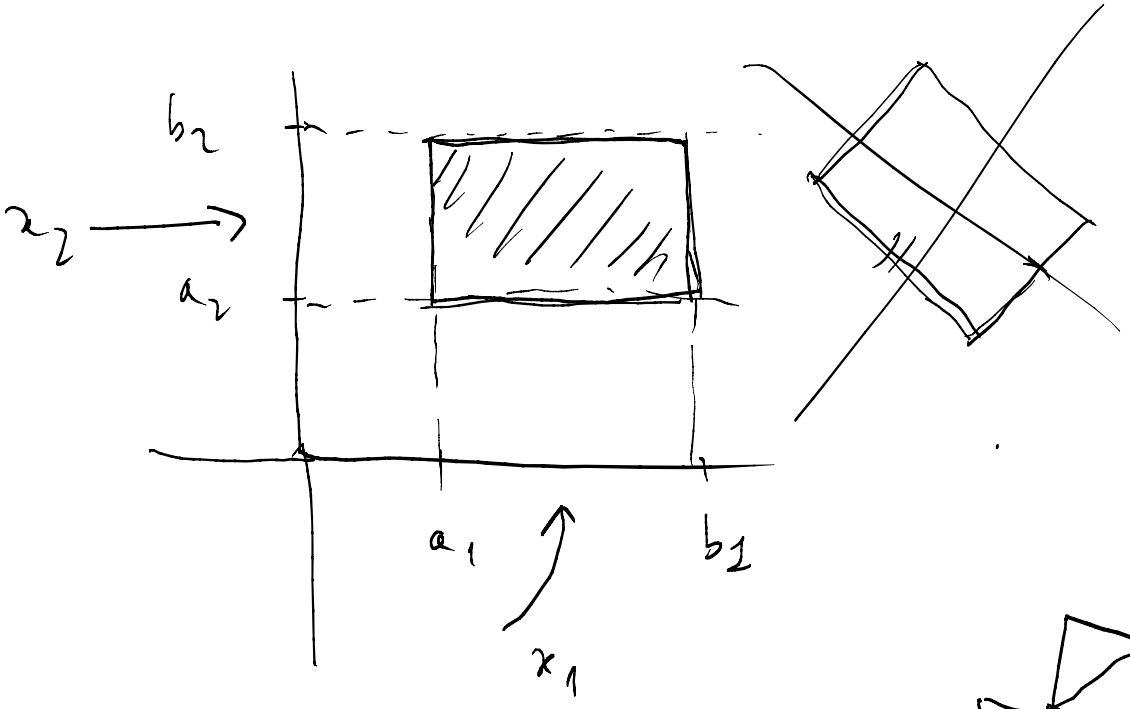
~ 1900

$$|\Omega| = \int_a^b f(t)\,dt$$

① INTERVALLI

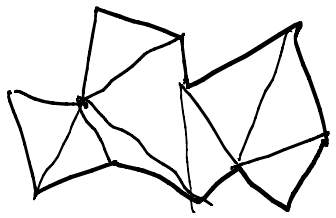
$$I = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n] =$$

$$= \{ (x_1, \dots, x_n) \in \mathbb{R}^n : \; x_i \in [a_i, b_i] \quad \forall i = 1 \dots n \}$$



$\mathbb{R}^3$

parallelepipedo
colle facce parallele
ai piani xy, xz, yz
$\{z=0\}$ $\{y=0\}$ $\{x=0\}$



Def $$|I| \equiv |b_1 - a_1| \times |b_2 - a_2| \times \dots \times |b_n - a_n| =$$

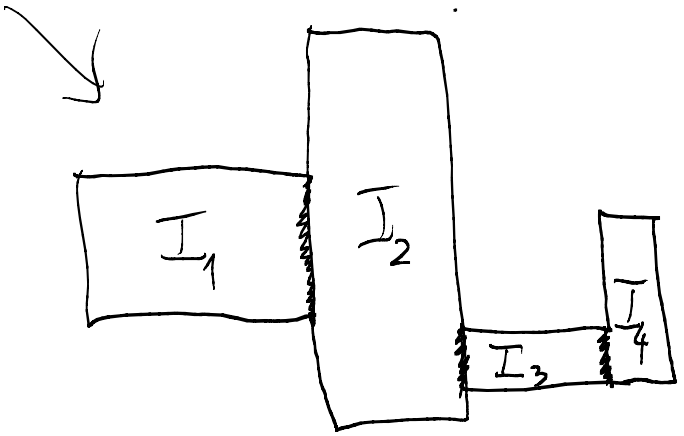
$$= \prod_{i=1..n} |b_i - a_i|$$

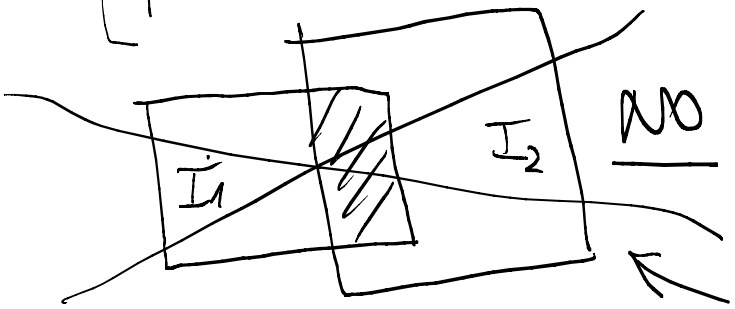
PLURINTERVALLI

$$P = \bigcup_{i=1}^{N} I_i$$

ove I_i è un intervallo $\forall i = 1..N$

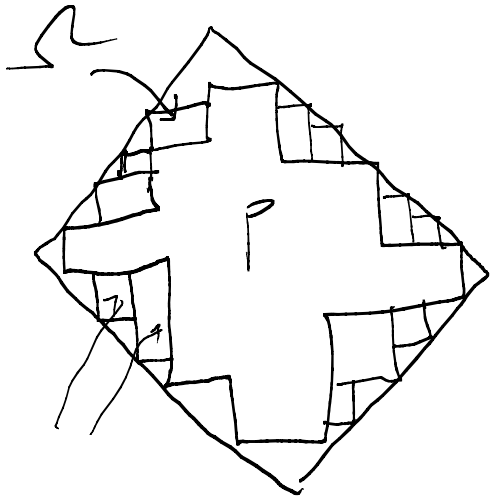
[verificanti $I_i \cap I_j$ non contiene punti interni di I_i e I_j

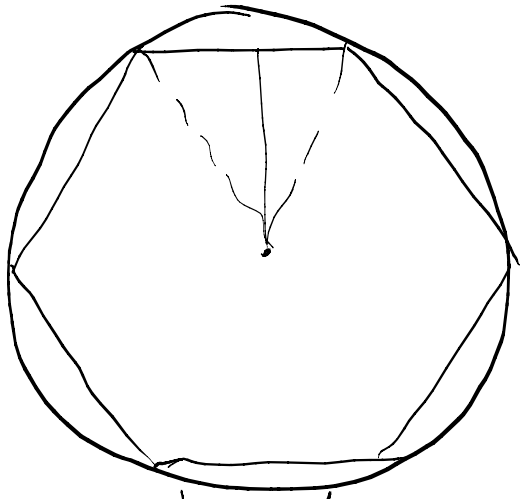




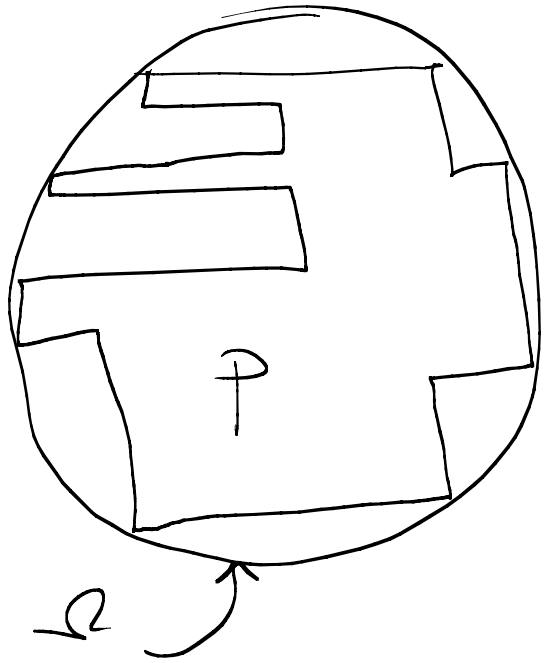
NO

$$|P| = \sum_{1}^{N} |I_i|$$





Archimede
(puntonte)



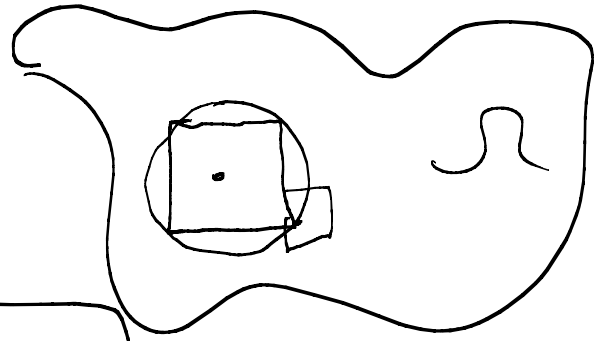
Se $\Omega \subset \Omega'$ allora $|\Omega| \leq |\Omega'|$

OCCORRE GARANTIRE

MONOTONIA DELLA MISURA

P contenuto in Ω se $P \subseteq \Omega$ $|P|$ è un'approssim. per difetto delle $|\Omega|$, anche sconosciuta.

Se Ω è APERTO



$\exists$ P plurirettangoli $P \subseteq \Omega$

e si pone

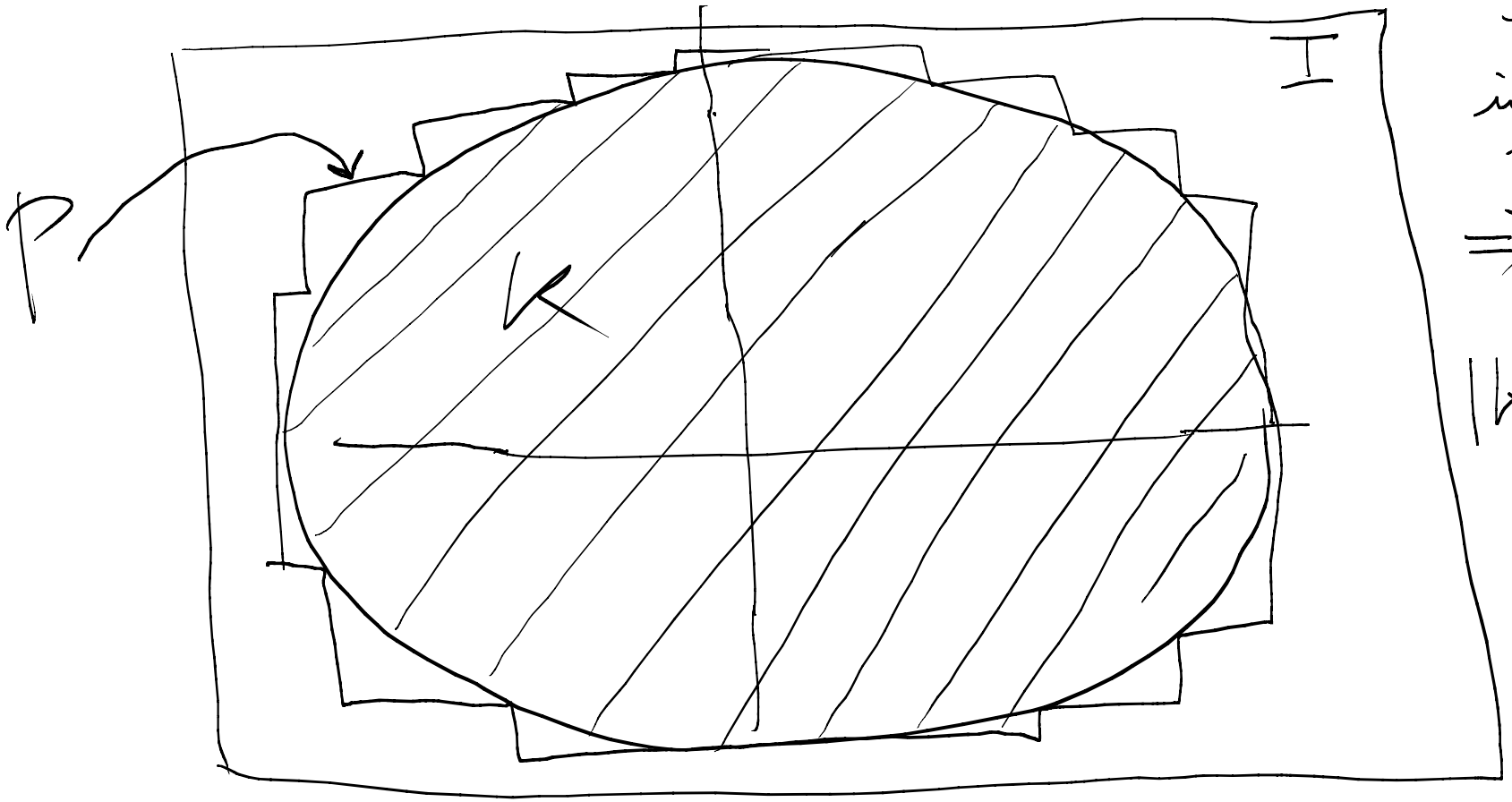
$$|\Omega| = \sup_{P \subseteq \Omega} |P|$$

DEF. di misura di un aperto

$\mathbb{R}^1$ $\quad \mathbb{Q} \subseteq \mathbb{R}^1$, ma non contiene intervalli.

Se K è compatto (chiuso e limitato)

Poiché K è limitato, è contenuto in una sfera



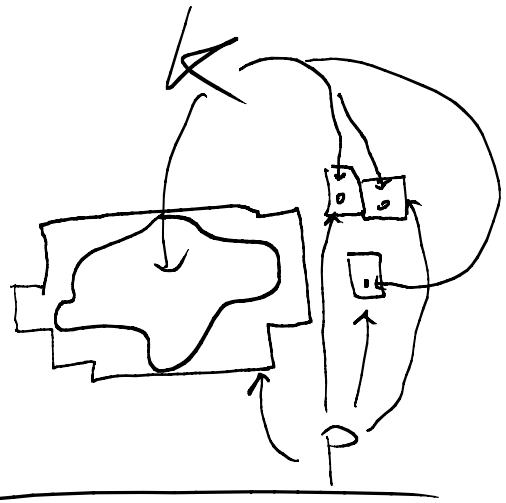
e quindi in un intervallo

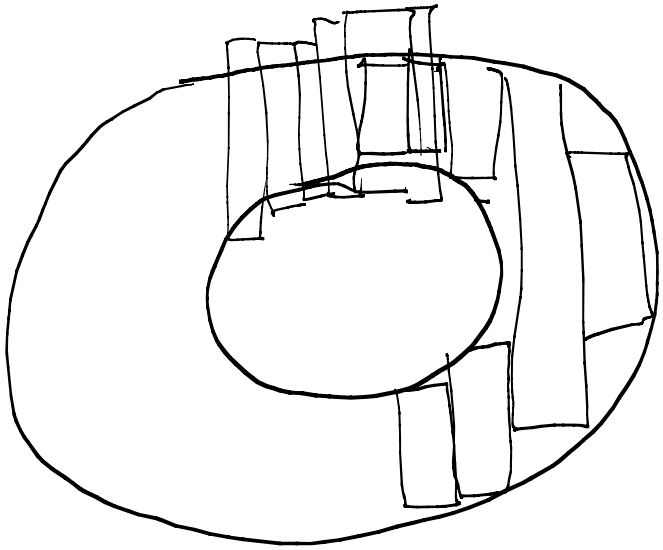
$\Rightarrow$ plurintervallo

$$|K| = \inf_{P \supseteq K} |P|$$

$$|\underset{\Omega}{\text{aperto}}| = \sup_{P \subseteq \Omega} |P|$$

$$|\underset{K}{\text{compatto}}| = \inf_{P \supseteq K} |P|$$





$E \subseteq \mathbb{R}^n$ sia un insieme limitato, arbitrario

$$|E|^* = \inf_{\substack{\Omega \supseteq E \\ \text{aperto}}} |\Omega|$$

MISURA ESTERNA di E
(approssimazione dall'esterno)

$$|E|_* = \sup_{\substack{K \subseteq E \\ K \text{ compatto}}} |K|$$

MISURA INTERNA di E
(approssim. dall'interno)

DEF. di misurabilità

E sarà detto MISURABILE (secondo LEBESGUE) se e solo se $|E|_* = |E|^*$ ed il valore comune sarà $|E|$

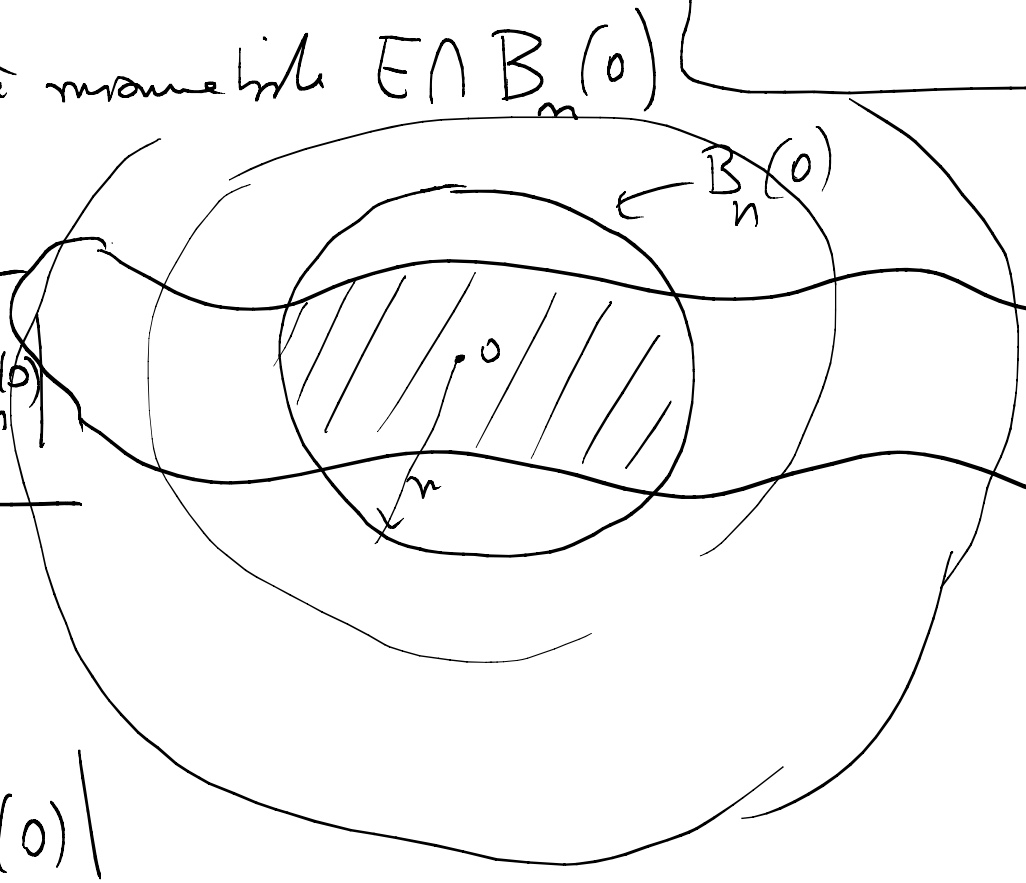
E NON LIMITATO

E si dirà numerabile se

$\forall n \in \mathbb{N}$ è numerabile $E \cap B_n(0)$

e infatti

$|E| = \sup |E \cap B_n(0)|$



$E \subseteq E'$ numerabile allora $|E| \leq |E'|$

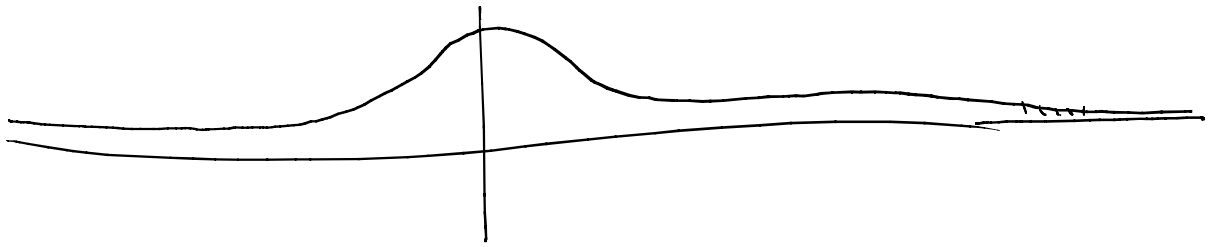
Oss.

$n \to |E \cap B_n(0)|$

è crescente, perché $B_n(0) \subseteq B_{n+1}(0) \Rightarrow E \cap B_n(0) \subseteq E \cap B_{n+1}(0)$

e per la monotonia della misura segue la tesi.

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx = \pi$$

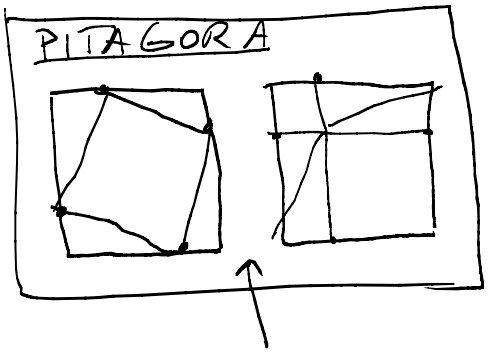


1) MONOTONIA $E \subseteq E'$, E, E' misur. $\Rightarrow |E| \leq |E'|$

2) SUBADDITIVITA' NUMERABILE E_i, $i \in \mathbb{N}$ insieme misurabile $\Rightarrow |\bigcup_1^\infty E_i| \leq \sum_1^\infty |E_i|$

3) ADDITIVITA' NUMERABILE E_i misurabili $|E_i \cap E_j| = 0 \Rightarrow |\bigcup_1^\infty E_i| = \sum_1^\infty |E_i|$

4) $E \subseteq E' \Rightarrow |E' \setminus E| = |E'| - |E|$



5) CONTINUITA' verso l'alto $E = \bigcup_1^\infty E_i$, E_i misurabili $E_i \subseteq E_{i+1}$ allora $|E| = \sup_{i=1,\infty} |E_i|$

6) " VERSO IL BASSO $E = \bigcap_1^\infty E_i$ $E_i \supseteq E_{i+1}$ $\boxed{|E_1| < +\infty} \Rightarrow |E| = \inf |E_i|$

$E_1 = \{x \in \mathbb{R} : x \geq 1\}$

$\vdots$

$E_n = \{x \in \mathbb{R} : x \geq n\}$

$\bigcap_1^\infty E_i = \emptyset$

$+\infty = |E_1| = |E_2| = \dots = |E_n| = \dots$

$\left|\bigcap_1^\infty E_i\right| = |\emptyset|$

$\emptyset \subseteq [-\varepsilon, \varepsilon]$

$|\emptyset| \leq 2\varepsilon \quad \forall \varepsilon > 0$

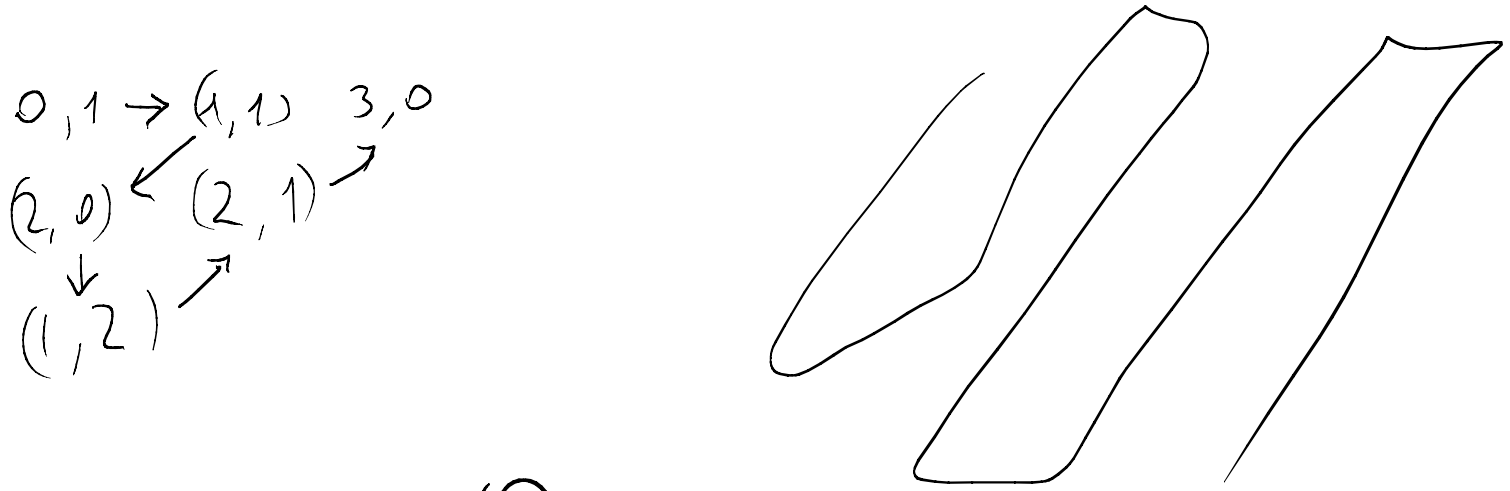
$\Downarrow$

$|\emptyset| = 0$

$E = \mathbb{Q} \cap [0,1]$

$q = \frac{m}{n}$

Q



ogni punto ha misura 0.

$$Q = \bigcup_{1}^{\infty} \text{punti} \qquad |Q| = \sum_{1}^{\infty} \underbrace{|\text{punti}|}_{\| \; 0} = 0$$