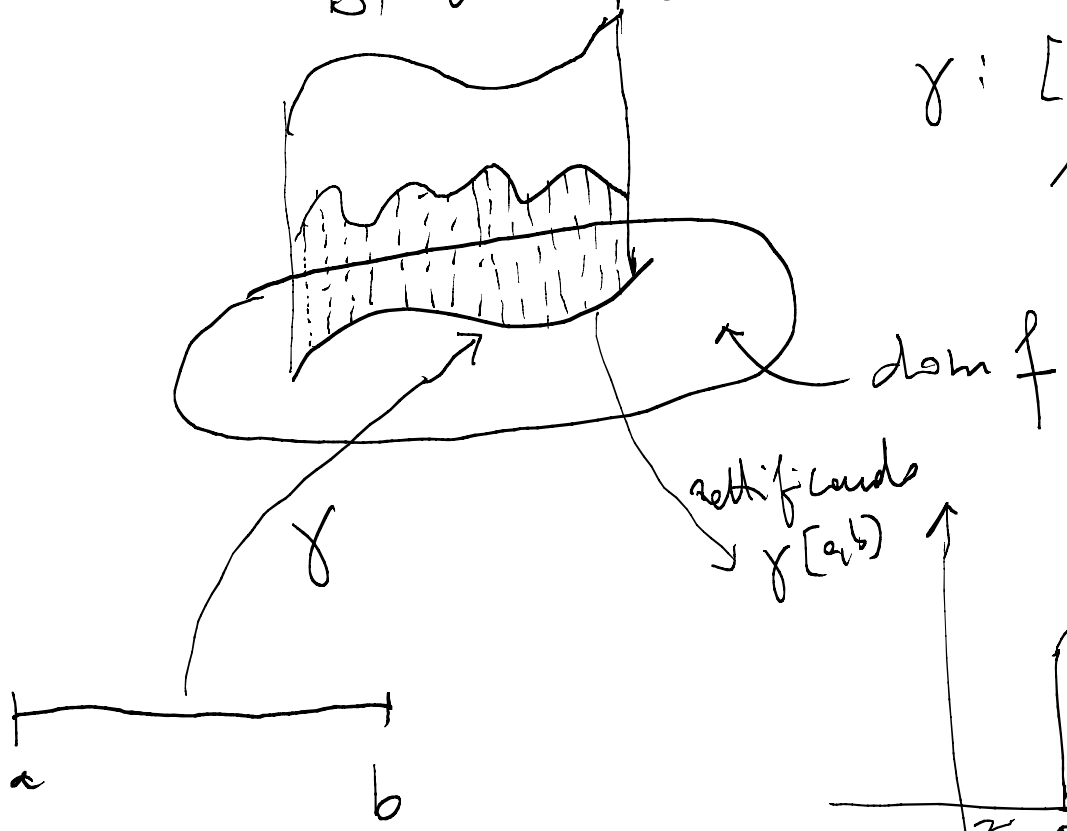
Note Title 5/19/2021

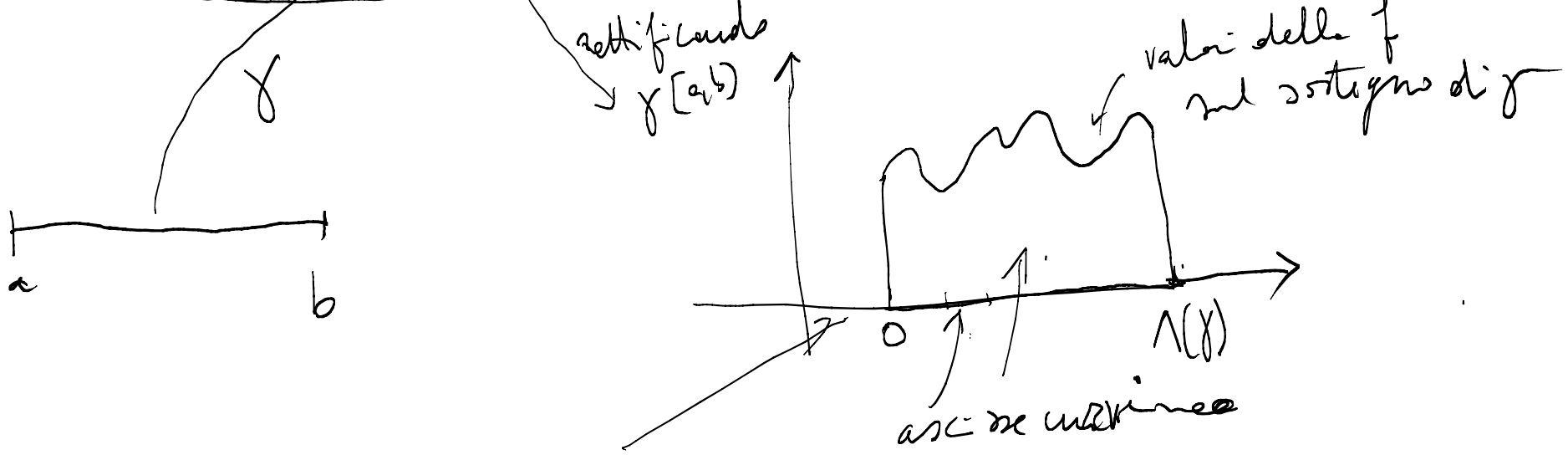
INTEGRALE CURVILINEO DI UNA FUNZIONE

$\gamma : [a,b] \to \Omega$

$f : \Omega \to \mathbb{R}$

$f \geq 0$





Si definisce INTEGRALE CURVILINEO di f su γ il numero

$$\int_\gamma f\,dl \equiv \int_a^b f(\gamma(t))\,|\dot\gamma(t)|\,dt$$

$f(\gamma(t)) \in \mathbb{R}$; $|\dot\gamma(t)| \in \mathbb{R}$; dl oppure ds

$$\Lambda(\gamma) = \int_\gamma 1\,dl$$

caso particolare di integrale curvilineo

$$\int_\gamma xz$$

$$\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix} \qquad t \in [0, 2\pi]$$

$$f(x,y,z) = xz \qquad f : \mathbb{R}^3 \to \mathbb{R}$$

$$\dot\gamma(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix}$$

$$\int_\gamma xz \equiv \int_0^{2\pi} \underbrace{f(x(t), y(t), z(t))}_{\cos t \cdot t} \cdot \underbrace{|\dot\gamma(t)|}_{\sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}}\,dt =$$

$x(t)$ $\quad$ $z(t)$

$$= \sqrt{2} \int_0^{2\pi} t\cos t\,dt = \sqrt{2}\left[t\sin t \Big|_0^{2\pi} - \int_0^{2\pi} 1\cdot \sin t\,dt \right] = \sqrt{2}\,(0-0)$$

$$= 0$$

Siano $\gamma(t)$ e $\sigma(s)$ parametrizzazioni equivalenti equivalenti di una curva

$\exists\, \rho : [c,d] \to [a,b]$ $\quad \gamma : [a,b] \to \Omega$

$\rho \in C^1$ invertibile strett. crescente: $\boxed{\dot\rho \geq 0}$ $\quad \sigma : [c,d] \to \Omega$

$f : \Omega \to \mathbb{R}$

$\to \quad \sigma(s) = \gamma(\rho(s))$

$\uparrow \quad \uparrow \quad t$

$$\int_\sigma f = \int_c^d f(\sigma(s))\,|\dot\sigma(s)|\,ds = \int_c^d f(\gamma(\rho(s)))\,|\underbrace{\dot\gamma(\rho(s))}_{\in \mathbb{R}^n}\,\underbrace{\dot\rho(s)}_{\in \mathbb{R}}|_{\mathbb{R}^n}\,ds =$$

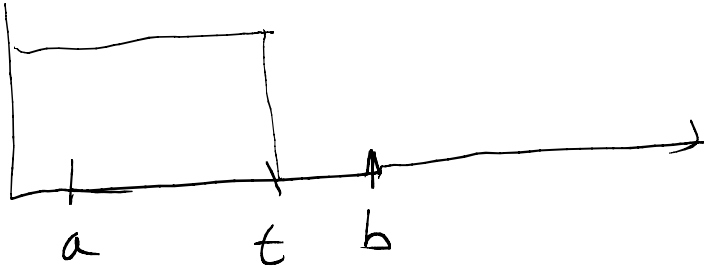
$$= \int_c^d f(\gamma(\rho(s)))\,|\dot\gamma(\rho(s))|\,\underbrace{\dot\rho(s)\,ds}_{dt} \overset{t=\rho(s)}{=\!=} \int_{\rho(c)=a}^{\rho(d)=b} f(\gamma(t))\,|\dot\gamma(t)|\,dt = \int_\gamma f$$

$\gamma : [a,b] \to \mathbb{R}^n$ regolare (C^1 e $\dot\gamma(t) \neq 0$ su tutto $[a,b]$)

$\Downarrow$

$|\dot\gamma(t)| > 0$

$$s(t) = \int_a^t |\dot\gamma(\tau)|\, d\tau$$



$s(t)$ = distanza percorsa sulla curva dal punto iniziale $\gamma(a)$

Ovviamente crescente

o meglio

$$s'(t) = |\dot\gamma(t)| > 0 \quad \forall t \in [a,b]$$

s è invertibile da $[a,b] \to [0, \Lambda(\gamma)]$

$\Lambda(\gamma)$ = lunghezza di γ su $[a,b]$

Denotando con $t(s)$ la funzione inversa da $[0, \Lambda(\gamma)] \to [a,b]$

Si può definire

$$\sigma(s) = \gamma(t(s)) \qquad s \in [0, \Lambda(\gamma)]$$

$t(s)$ = tempo necessario a percorrere la distanza s.

$$\sigma'(s) = \dot\gamma(t(s))\, t'(s) = \qquad t \in [0, \Lambda(\gamma)]$$

$$= \dot\gamma(t(s)) \cdot \underbrace{s'(t(s))}_{\| \; |\dot\gamma(t(s))|}$$

$t(s)$ = l'inverso di $s(t)$

$$t'(s) = \frac{1}{s'(t(s))}$$

in definitiva $\sigma'(s) = \dfrac{\dot\gamma(t(s))}{|\dot\gamma(t(s))|}$ ← VERSORE

$$\Rightarrow \boxed{|\sigma'(s)| \equiv 1 \quad \forall\, s \in [0, \ell(\gamma)]}$$

$$|\sigma'(s)| \equiv 1 \Rightarrow \frac{d}{ds}|\sigma'(s)| \equiv 0 \quad \text{su } [0, \ell(\gamma)]$$

$$0 = \frac{d}{ds}\sqrt{\sigma'(s)\cdot\sigma'(s)} = \frac{1}{2\sqrt{\sigma'(s)\cdot\sigma'(s)}}\,[2\,\sigma'(s)\,\sigma''(s)] =$$

$$\sqrt{\sigma'(s)\cdot\sigma'(s)} = 1$$

$$= \sigma'(s)\,\sigma''(s)$$

quindi σ' (vettore tangente standard) è ortogonale a $\sigma''(s)$, che verrà detto vettore normale standard (formula di FRENET (frenè))

$$\sigma''(s) = \underbrace{|\sigma''(s)|}_{\substack{\text{CURVATURA} \\ \text{di } \sigma \text{ in } \sigma(s)}} \frac{\sigma''(s)}{|\sigma''(s)|}$$ versore normale alla curva

CASO PARTICOLARE IMPORTANTE

$$\underset{R>0}{\gamma(t)} = \begin{pmatrix} R\cos t \\ R\sin t \end{pmatrix} \quad t \in [0, 2\pi] \qquad \begin{matrix} a = 0 \\ b = 2\pi \end{matrix}$$

$$\dot\gamma(t) = \begin{pmatrix} -R\sin t \\ R\cos t \end{pmatrix}$$

$$|\dot\gamma(t)| = R$$

$$s(t) = \int_0^t |\dot\gamma(\tau)|\, d\tau = \int_0^t \sqrt{R^2\sin^2\tau + R^2\cos^2\tau}\, d\tau =$$

$$= R\int_0^t d\tau = Rt$$

$$s : [0, 2\pi] \to [0, 2R\pi]$$

$$\boxed{t(s) = \frac{s}{R}}$$

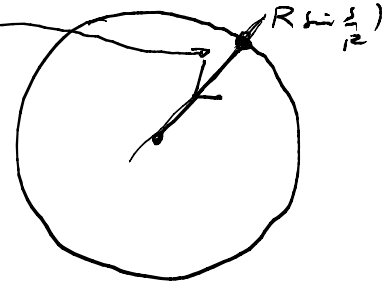
$$\sigma : [0, 2R\pi] \to \mathbb{R}^2$$

$$\sigma(s) = \gamma(t(s)) = \gamma\left(\frac{s}{R}\right)$$

$$\sigma'(s) = \begin{pmatrix} -R\sin\left(\frac{s}{R}\right)\cdot\frac{1}{R} \\ R\cos\left(\frac{s}{R}\right)\cdot\frac{1}{R} \end{pmatrix} = \begin{pmatrix} -\sin\left(\frac{s}{R}\right) \\ \cos\left(\frac{s}{R}\right) \end{pmatrix}$$

$$\sigma(s) = \begin{pmatrix} \cos\left(\frac{s}{R}\right) \\ \sin\left(\frac{s}{R}\right) \end{pmatrix} \qquad s \in [0, 2R\pi]$$

$$\sigma''(s) = \begin{pmatrix} -\cos\left(\frac{s}{R}\right) & \frac{1}{R} \\ -\sin\left(\frac{s}{R}\right) & \frac{1}{R} \end{pmatrix} = \frac{1}{R} \begin{pmatrix} \cos\left(\frac{s}{R}\right) \\ \sin\left(\frac{s}{R}\right) \end{pmatrix}$$



curvature

Raggio di curvatura

$$\text{Raggio di curvatura} = \frac{1}{\text{curvature}} =$$

$$= \frac{1}{|\sigma''(s)|}$$

GEOMETRIA DIFFERENZIALE