Note Title 5/12/2021

$A \in C^0$ integrabile in Ω, $\gamma, \sigma : [0,1] \to \Omega$ (C^1)

$\int_\gamma A = \int_\sigma A$ se $\gamma(0) = \sigma(0)$ $\gamma(1) = \sigma(1)$

$\gamma \sim \sigma \Leftrightarrow$ $f(\gamma(1)) - f(\gamma(0))$ dove f è potenziale $\nabla f \equiv A$ in Ω

DIM

- $\exists f : \nabla f \equiv A$ in Ω $\quad \gamma \in C^1[0,1]$

$$\int_\gamma A \equiv \int_0^1 \underbrace{A(\gamma(t))\dot\gamma(t)}_{\text{continua da } [0,1] \text{ in } \mathbb{R}}\, dt = \int_0^1 \underbrace{\nabla f(\gamma(t))\,\dot\gamma(t)}_{\text{continua da } [0,1] \to \mathbb{R}}\, dt =$$

$$= \int_0^1 \frac{d}{dt}\left[f(\gamma(t))\right] dt \overset{\text{Torr.}}{=} f(\gamma(1)) - f(\gamma(0))$$

$$\nabla f(\gamma(t))\,\dot\gamma(t) = \frac{d}{dt} f(\gamma(t))$$

∎

$$\frac{d}{dt}\left[f(\gamma(t))\right] = \nabla f(\gamma(t)) \cdot \dot{\gamma}(t)$$

$\nabla f(\gamma(t)) \in \mathbb{R}^n$, $\dot{\gamma}(t) \in \mathbb{R}^n$

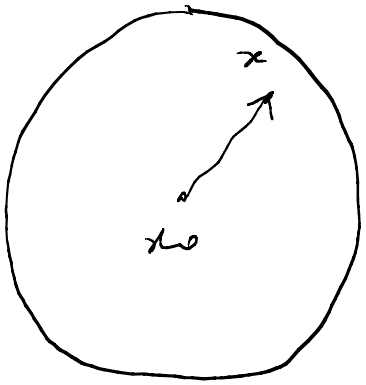
produit scalaire

$\gamma : [0,1] \to \Omega$

$f : \Omega \to \mathbb{R}$

$$D\left[f(\gamma(t))\right] = Df(\gamma(t))\, D\gamma$$

$$\boxed{\left[f(\gamma(t))\right]' = f'(\gamma(t)) \cdot \gamma'(t)} \longleftarrow$$



$$h(t) = f(x_0 + t(x - x_0))$$

$$h(0) = f(x_0)$$

$$h(1) = f(x)$$

$f \in C^1$

$$h'(t) = \nabla f\big(x_0 + t(x - x_0)\big)(x - x_0)$$

$\exists\, \xi \in [0,1]$:

$$h(1) - h(0) = (1-0)\, h'(\xi) \leftarrow \text{Th. Lagrange}$$

$$\boxed{f(x) - f(x_0) = \nabla f\big(x_0 + \xi(x - x_0)\big)(x - x_0)}$$

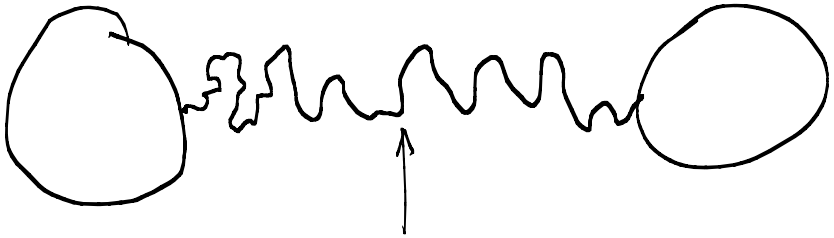
$x^3y - x = 1$ contiene punti (non vuoto)

$y = 0 \Rightarrow x = -1 \quad (-1, 0) \in \{x^3y - x = 1\}$

$$\begin{cases} f_x = 3x^2y - 1 = 0 \\ f_y = x^3 = 0 \end{cases}$$

$\nabla f = 0$ se $x = 0 \Rightarrow -1 = 0$ N. Sol

$\nabla f \neq 0 \Rightarrow$ almeno una delle incognite può essere espressa come funzione dell'altra



$\gamma \in C^1$

$w = \rho e^{i\theta}$

$$\begin{cases} \alpha_k w^k \in \mathbb{R} \quad < 0 \quad (\alpha_k \neq 0) \\ |\alpha_k w^k| < 1 \end{cases}$$

$|a+b| = |a| - |b|$

$|b| < |a|$

$b < 0$

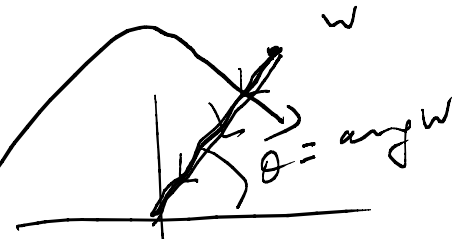
$a > 0$

$$q(w) = \underbrace{1 + \alpha_k w^k}_{} + \underbrace{w^{k+1} \tilde{q}(w)}_{}$$

$$|q(w)| = \underbrace{|1 + \alpha_k w^k|}_{\substack{>0 \\ <0 \\ |\cdot|=1 \quad | \quad |<1}} + |w|^{k+1} |\tilde{q}(w)| = 1 - \underbrace{|\alpha_k| |w|^k + |w|^{k+1} |\tilde{q}(w)|}_{=}$$

$$= 1 - |w|^k \left[\underbrace{|\alpha_k|}_{>0} + \underbrace{|w| |\tilde{q}(w)|}_{} \right]$$

$$[\quad] \to |\alpha_k| > 0 \quad \text{per } w \to 0$$



$$\arg(\alpha_k w^k) = \pi$$

$$\boxed{\arg w = \frac{\pi - \arg \alpha_k}{k}}$$

Perm. segno $\Rightarrow$ $[\quad] > 0$ per $|w| < \delta$

$$|q(w)| = 1 - \underbrace{|w|^k}_{\substack{>0 \\ w \neq 0}} \underbrace{[\quad]}_{>0} < 1 \text{ per } |w| < \delta$$

$$< \frac{1}{|\alpha_k|^{\frac{1}{k}}}$$

$$\boxed{q(w) = \frac{p(z_0 + w)}{p(z_0)}}$$

$$1 > |q(w)| = \frac{|p(z_0+w)|}{|p(z_0)|}$$

Ω stella

$\rightsquigarrow$ fatto $\rightarrow x_0 = 0$

$\Updownarrow$

$x \in \Omega \qquad \overrightarrow{x_0 x} \subseteq \Omega$

γ chiusa in Ω

$$h(\lambda, t) = \boxed{(1-\lambda)\,\gamma(t)} \in \overrightarrow{0\,\gamma(t)}$$

$\gamma(t) \in \Omega \Rightarrow$ tutto il segmento $\overrightarrow{0\,\gamma(t)} \subseteq \Omega$

$$h(0, t) = \gamma(t)$$

$$h(1, t) = 0 \cdot \gamma(t) = 0 \in \mathbb{R}^n$$

$x_0 \neq 0$

$$\begin{cases} h(\lambda, t) = x_0 + (1-\lambda)\left[\gamma(t) - x_0\right] \\ h(0, t) = x_0 + 1\left[\gamma(t) - x_0\right] = \gamma(t) \\ h(1, t) = x_0 \end{cases}$$

segmento di estremi $x_0 \in \gamma(t) \subseteq \Omega$

$$\phi(u,v) = \begin{pmatrix} u^2 v \\ u^2 v^3 \\ 1 + uv \end{pmatrix} \qquad (1,1,2)$$

$$\begin{cases} u^2 v = 1 \\ u^2 v^3 = 1 \\ 1 + uv = 2 \end{cases} \xrightarrow{\;u\;} \begin{cases} u^2 \frac{1}{u} = 1 \Rightarrow u = 1 \\ v^3 = 1 \Rightarrow v = 1 \end{cases}$$

$$\Downarrow$$

$$uv = 1$$

$$\Uparrow$$

$$\begin{cases} u \neq 0 \Rightarrow v = \frac{1}{u} \\ v \neq 0 \Rightarrow u = \frac{1}{v} \end{cases}$$

$u = 0$ oppure $v = 0$ sono da scartare perché incompatibili con la prima delle equazioni

$$\phi(u,v) = \begin{pmatrix} u^2 v \\ u^2 v^3 \\ 1 + uv \end{pmatrix}$$

$$\rightarrow \quad \phi'(u,v) = \begin{pmatrix} 2uv & u^2 \\ 2uv^3 & 3u^2v^2 \\ v & u \end{pmatrix}$$

$$\phi'(1,1) = \begin{pmatrix} 2 & 1 \\ 2 & 3 \\ 1 & 1 \end{pmatrix}$$

Parametric

$$\psi(\alpha,\beta) = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

$$\nu(1,1) = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix}$$

$$-(x-1) - (y-1) + 4(z-2) = 0$$

$$\boxed{4z - x - y = 6}$$ IMPLIC

$$f(x,y)=0$$

$\exists\ \varphi$:

$$f(x,\varphi(x)) \equiv 0 \qquad I_\delta(x_0)$$

$\Rightarrow$ $y=\varphi(x)$ è la formula

risolutiva dell'eq.

$\Rightarrow$ $f(x,y)=0$

implicita

1) $f: \mathbb{R} \to \mathbb{R}$

2) $f: \mathbb{R}^n \to \mathbb{R}$

codom. Scalare

$$\begin{pmatrix} -f'(x_0) \\ 1 \end{pmatrix} \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} = 0$$

1) $$\boxed{y = \underbrace{f(x_0)}_{y_0} + f'(x_0)(x-x_0)}$$

2) $$y = \underbrace{f(x_0)}_{y_0} + \nabla f(x_0)(x-x_0)$$

$$\boxed{\sum_{1}^{n} f_{x_i}(x_0)(x-x_0)_i}$$

equazioni esplicite (o implica portando tutto a I membro)

$$-f_{x_1}(x_0)(x-x_0)_1 - \dots - f_{x_n}(x_0)(x-x_0)_n + y - y_0 = 0$$

vett. normale $\begin{pmatrix} -\nabla f \\ 1 \end{pmatrix} \begin{pmatrix} (x-x_0)_1 \\ \vdots \\ (x-x_0)_n \\ y-y_0 \end{pmatrix} = 0$

spost. sul piano tangente

$f : \mathbb{R} \to \mathbb{R}^n$

$f : \mathbb{R}^m \to \mathbb{R}^n$

codom. vettoriale

curve

$$\gamma(t) = \underbrace{f(t_0)}_{\in \mathbb{R}^n} + \underbrace{f'(t_0)}_{\in \mathbb{R}^n} \underbrace{t}_{\in \mathbb{R}}$$

eq. parametrica retta param. per $f(t_0)$ tangente al sostegno

$$\psi(w) = f(w_0) + f'(w_0)\, w \in \mathbb{R}^n = f(w_0) + \sum_{i=1}^{m} w_i \underbrace{f_{w_i}(w_0)}_{\text{colonne } i\text{-esime di } f'}$$

$w \in \mathbb{R}^m$, $f'(w_0) \in \mathbb{R}^{n \times m}$, $(t - t_0) \in \mathbb{R}$

$\exists w: \quad |p(z_0+w)| < |p(z_0)| \qquad \triangleright$ grado $n \geq 0$

z'

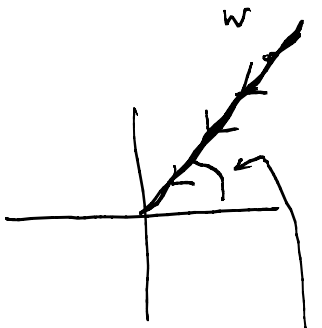
$q(w) = \dfrac{p(z_0+w)}{p(z_0)}$ polinomio dello stesso grado

$0 \neq p(z_0)$

$q(w) = 1 + \alpha_k w^k + w^{k+1}\tilde{q}(w)$

$\uparrow$ primo $k>0$ $\alpha_k \neq 0$

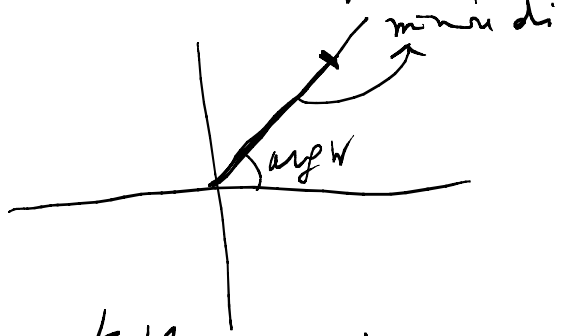
$|q(w)| \leq |1 + \alpha_k w^k| + |w|^{k+1}\tilde{q}(w)$



Scegliamo w in modo da avere $\alpha_k w^k$ reale e negativo

$\pi = \arg \alpha_k w^k = \arg \alpha_k + k \arg w \qquad \arg w = \dfrac{\pi - \arg \alpha_k}{k}$ $k>0$

$$|\alpha_k w^k| < 1 \qquad |w|^k < \frac{1}{|\alpha_k|} \Rightarrow |w| < \frac{1}{\sqrt[k]{|\alpha_k|}}$$



$$\rightarrow \Big|\underset{a}{1} + \underbrace{\alpha_k w^k}_{b}\Big| = 1 - |\alpha_k||w|^k$$

$$|a+b| = |a| - |b|$$

$a>0 \quad b<0 \quad |b|<|a|$

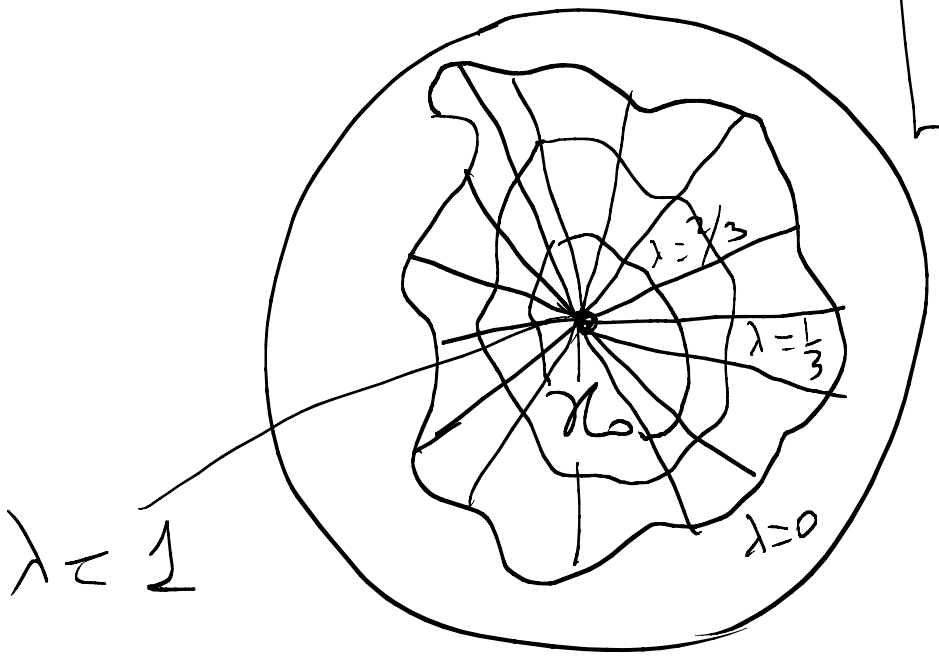
$$\Downarrow$$

$$|q(w)| \le 1 - |\alpha_k||w|^k + |w|^{k+1}|\tilde{q}(w)| =$$

$$= 1 - \underbrace{|w|^k}_{>0}\underbrace{\Big[\underbrace{|\alpha_k|}_{\substack{>0 \\ \text{perché} \\ \alpha_k \neq 0}} + |w|\,|\tilde{q}(w)|\Big]}_{\substack{\downarrow\, w\to 0 \text{ sulla semiretta} \\ |\alpha_k|}}$$

Per la perm. segno $\exists \delta$: $[\quad] > 0$ per $|w| < \delta$

$$|w| < \min\left\{\delta, \frac{1}{|\alpha_k|^{\frac{1}{k}}}\right\} \Rightarrow \boxed{|q(w)| < 1} \quad \boxed{\text{Tesi}}$$



γ chiusa in Ω

$\gamma : [0,1] \to \Omega \quad \gamma(0) = \gamma(1)$

"gli segni è contenuta in Ω"

$$h : [0,1] \times [0,1] \to \Omega$$

$$h(0,t) = \gamma(t)$$

$$h(1,t) = x_0$$

$\sigma(t) \equiv x_0$
$\forall t \in [0,1]$

$$h(\lambda,t) = x_0 + (1-\lambda)[\gamma(t) - x_0]$$

$\lambda \in [0,1]$ segment di estremi x_0 e $\gamma(t)$

$$h(0,t) = x_0 + \gamma(t) - x_0 = \gamma(t)$$

$$h(1,t) = x_0 + 0\,[\gamma(t) - x_0] = x_0$$

se $x_0 = 0$ $\quad h(\lambda,t) = (1-\lambda)\gamma(t)$

$$\boxed{Df(\gamma(t))\,\dot\gamma(t)} = \frac{d}{dt}\left[f(\gamma(t))\right]$$

↑ ocorre che f sia diff! È vero perché $f \in C^1$

$$\int_a^b \underbrace{\frac{d}{dt}\left[f(\gamma(t))\right]}_{\in C^0} = f(\gamma(b)) - f(\gamma(a))$$

↓ Torricelli

$\in C^0$

$$\int_a^b f'(t)\,dt = f(b) - f(a)$$

$f' \in C^0$

$$f(x,y) = \begin{cases} 0 & (0,0) \\ \dfrac{x^3 - y^4}{x^2+y^2} & (x,y) \neq (0,0) \end{cases}$$

3-om $\quad$ 4-om.

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} - \frac{y^4}{x^2+y^2} = 0 = f(0,0)$$

2-omog. $\neq 0$ salvo che in $(0,0)$

f è continua

$$\lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^3}{h^2+0}}{h} = 1$$

$$\lim_{k\to 0} \frac{1}{k}\left[f(0,k)-f(0,0)\right] =$$

$$= \lim_{k\to 0} \frac{1}{k}\left[\frac{k^4}{0+k^2}\right] = 0$$

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) \overset{f(0,0)}{-0} \overset{\text{def.}}{-0}}{\sqrt{h^2+k^2}} =$$

$$= \lim_{h,k\to 0} \frac{\frac{h^3-k^4}{h^2+k^2}}{\sqrt{h^2+k^2}} = \lim_{(0,0)} \boxed{\frac{h^3-k^4}{\boxed{(h^2+k^2)^{\frac{3}{2}}}\ 3}}$$

0-omog.

4 metri il resto e 1-omog.

$x_n : \underline{|x_n| = 1}$ $\forall n$: $\underline{|f(x_n)| > n}$

$\forall \varepsilon > 0 \ \exists \delta > 0 : |x| < \delta \quad \boxed{|f(x)| < \varepsilon}$

$\left|\frac{\delta}{2} x_n\right| = \frac{\delta}{2} \Rightarrow \boxed{y_n = \frac{\delta}{2} x_n} \in \underline{B_\delta(0)}$ $\forall n$

$\left|f\left(\frac{\delta}{2} x_n\right)\right| = \left|\left(\frac{\delta}{2}\right)^\alpha f(x_n)\right| = \left(\frac{\delta}{2}\right)^\alpha |f(x_n)| >$

$> \underbrace{\left(\frac{\delta}{2}\right)^\alpha n} \geq \varepsilon$ per n grande

$n > \varepsilon \left(\frac{2}{\delta}\right)^\alpha$