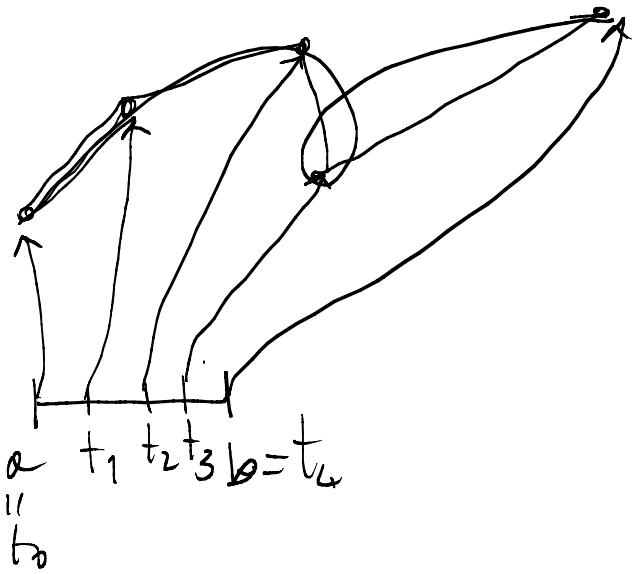
Note Title 5/12/2021

$$\gamma : [a,b] \to \mathbb{R}^n$$

① Π (partizione) $= a = t_0 < t_1 < t_2 < \dots < t_{n-1} < t_n = b$

② $$\Lambda(\Pi) \equiv \sum_{0}^{n-1} |\gamma(t_{i+1}) - \gamma(t_i)|$$

Lunghezza della poligonale inscritta associata alla partizione Π



③ $$\Lambda(\gamma) \equiv \sup_{\Pi} \Lambda(\Pi)$$

Se $\Lambda(\gamma) < \infty$ allora γ si dice RETTIFICABILE e $\Lambda(\gamma)$ sarà la sua lunghezza

altrimenti ($\Lambda(\gamma) = +\infty$) la γ sarà detta NON rettificabile

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$$\int \gamma(t)\,dt \equiv \begin{pmatrix} \int \gamma_1(t)\,dt \\ \int \gamma_2(t)\,dt \\ \vdots \\ \int \gamma_n(t)\,dt \end{pmatrix}$$

Lemma $a < b$

$$\left| \underbrace{\int_a^b \gamma(t)\,dt}_{\text{integrale di un vettore}} \right|_{\mathbb{R}^n} \le \underbrace{\int_a^b \underbrace{|\gamma(t)|_{\mathbb{R}^n}}_{\in \mathbb{R}}\,dt}_{\text{integrale di una funzione scalare}}$$

"DISUGUAGLIANZA TRIANG" $\qquad \int \approx \Sigma$

"PSEUDO-DIMOSTRAZIONE"

$$\left| \int_a^b \gamma(t)\,dt \right| \approx \left| \sum_0^{n-1} \gamma(t_i)(t_{i+1} - t_i) \right| \le \sum_0^{n-1} |\gamma(t_i)(t_{i+1} - t_i)| = \sum_0^{n-1} (t_{i+1} - t_i)|\gamma(t_i)| \approx \int_a^b |\gamma(t)|\,dt$$

↑ DISUG. TRIANGOLARE VERA ($\mathbb{R}^N$)

Th rettificabilità delle curve $C^1[a,b]$

$\gamma : [a,b] \to \mathbb{R}^N$ di classe $C^1[a,b]$

Ts Allora γ è rettificabile e, inoltre, $\Lambda(\gamma) \leq \int_a^b |\dot\gamma(t)|\,dt$

Dim. Scelta $\pi = a < t_0 < t_1 < \dots < t_n$

Si osserva che

$$\gamma(t_{i+1}) - \gamma(t_i) = \int_{t_i}^{t_{i+1}} \dot\gamma(t)\,dt \qquad \text{Ter. Torricelli}$$

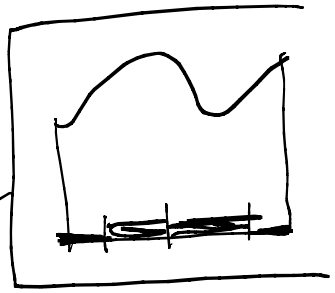
$$\Downarrow$$

$$|\gamma(t_{i+1}) - \gamma(t_i)| = \left| \int_{t_i}^{t_{i+1}} \dot\gamma(t)\,dt \right| \leq \int_{t_i}^{t_{i+1}} |\dot\gamma(t)|\,dt$$

Lemma precedente

$$\Lambda(\pi) = \sum_0^{n-1} |\gamma(t_{i+1}) - \gamma(t_i)| \leq \sum_0^{n-1} \int_{t_i}^{t_{i+1}} |\dot\gamma(t)|\,dt = \boxed{\int_a^b |\dot\gamma(t)|\,dt} \leftarrow \text{indipendente dalla partizione } \pi$$

MAGGIORANTE $\Lambda(\pi)$ $\forall \pi$



$\Rightarrow \quad \Lambda(\gamma) \quad \leq \quad \int_a^b |\dot\gamma(t)|\,dt$

$\sup \Lambda(\Pi) =$ MINIMO DEI MAGGIORANTI $\Lambda(\overline{\Pi})$ $\forall \overline{\Pi}$

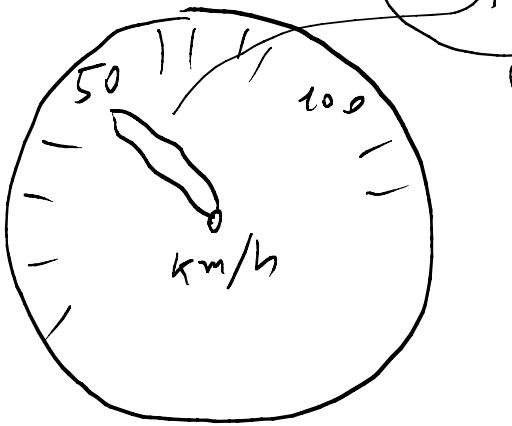
$\int_a^b |\dot\gamma(t)|\,dt$ UN MAGGIORANTE $\Lambda(\overline{\Pi})$ $\forall\, \overline{\Pi}$

Ne segue $\Lambda(\gamma) \leq \int_a^b |\dot\gamma(t)|\,dt$ ∎

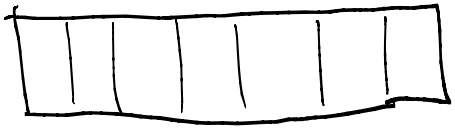
INOLTRE

Th (senza dimostr.) $\quad \boxed{\Lambda(\gamma) = \int_a^b |\dot\gamma(t)|\,dt}$

$\int |\dot{\gamma}(t)|\,dt$?



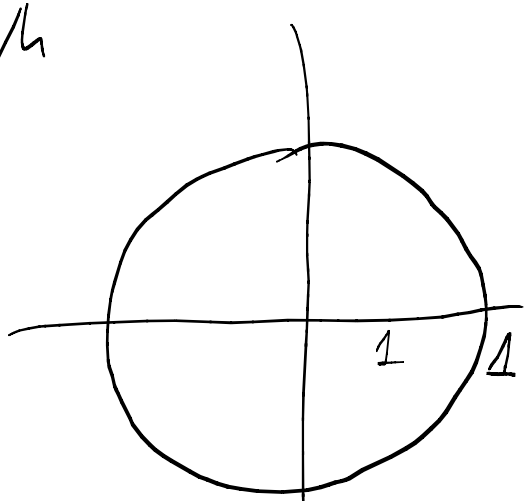
km



1 h a $\underbrace{100\text{km/h}}_{|\dot{\gamma}|}$ =

= $\underbrace{100\,\text{km}}$

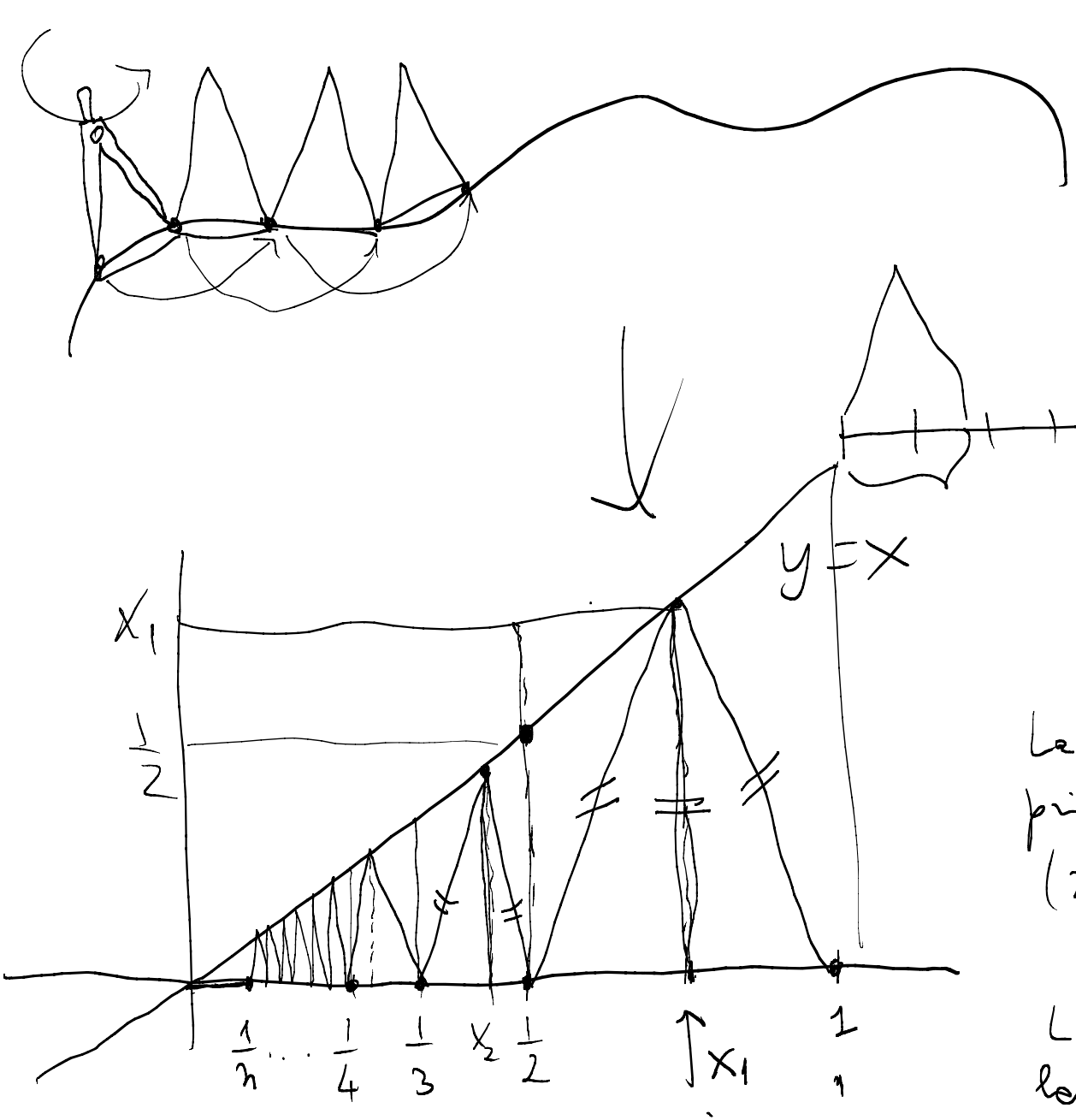
$\int_{1\text{h}} 100\text{km/h}$

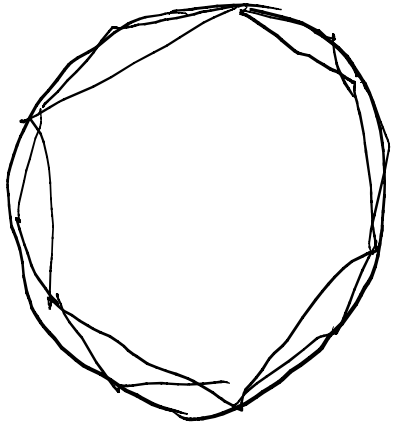


$(\cos 2t, \sin 2t)$ $t \in [0, 2\pi]$

$(\cos t, \sin t)$ $t \in [0, 2\pi]$

$(\cos t, \sin t)$ $t \in [0, 4\pi]$





$$\gamma(t) = \begin{pmatrix} t \\ f(t) \end{pmatrix}$$

La somma dei due lati del primo triangolo è $> 2 \cdot \frac{1}{2}$
(2 volte l'altezza per $x=1$ che è maggiore di quella effettiva.

La somma dei successivi due lati è $> 2 x_2 > 2 \frac{1}{3}$

Lunghezza della spezzata $\geq$ n triangoli + tratto finale

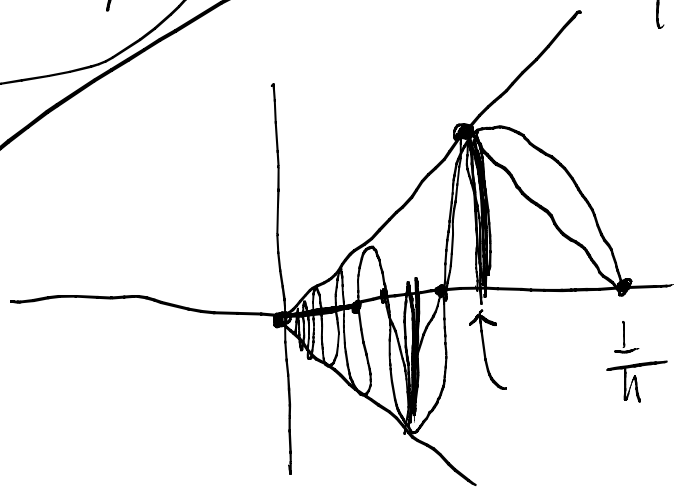
$$> 2\cdot\frac{1}{2} + 2\cdot\frac{1}{3} + 2\cdot\frac{1}{4} + \dots + 2\frac{1}{n} =$$

$$= \boxed{2\sum_{k=2}^{n}\frac{1}{k}} > \boxed{\sum_{k=2}^{n}\frac{1}{k}} \longrightarrow +\infty$$

$$\gamma(t) = \begin{pmatrix} t \\ f(t) \end{pmatrix} \qquad f(t) = \begin{cases} 0 & t = 0 \\ t \sin\frac{1}{t} & t \in \left]0, \frac{1}{\pi}\right] \end{cases}$$

$$\sin\frac{1}{t}\Big|_{t=\frac{1}{n\pi}} = \sin n\pi = 0$$

$\longrightarrow$



Continua in $\left[0, \frac{1}{\pi}\right]$

$\sin\frac{1}{t} = 0$ se $\frac{1}{t} = k\pi$ $\quad t = \frac{1}{k\pi}$ $\quad k \in \mathbb{N}$

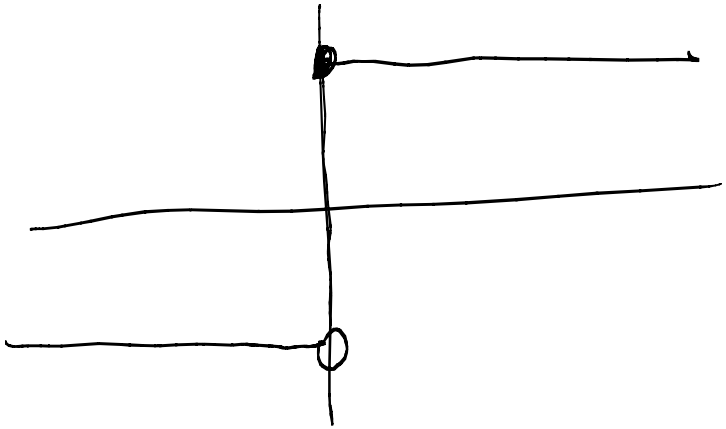
$$\sin\frac{1}{t} = 1 \qquad \frac{1}{t} = \frac{\pi}{2} + 2k\pi$$

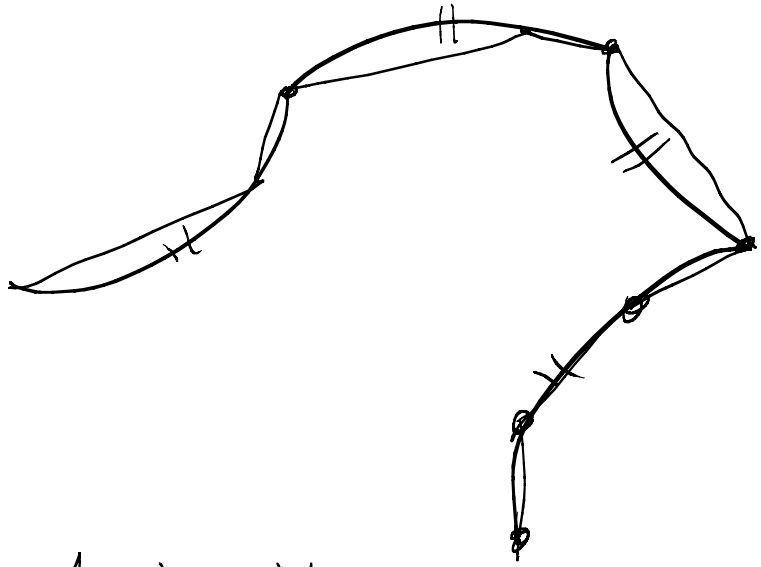
$$f\left(t = f\left(\frac{1}{\frac{\pi}{2} + 2k\pi}\right)\right) = \frac{1}{\frac{\pi}{2} + 2k\pi}$$

$$\sin\frac{1}{t} = -1 \qquad \frac{1}{t} = -\frac{\pi}{2} + 2k\pi$$

$$\Lambda(\text{Polygon}) \geq \sum\left(\frac{1}{\frac{\pi}{2} + 2k\pi}\right) + \left(\sum\left(\frac{1}{-\frac{\pi}{2} + 2k\pi}\right)\right)$$

$$\simeq \frac{1}{k}$$





C'est tout