Note Title 5/5/2021

CN A campo $C^0(\Omega)$, Ω aperto, integrabile $\gamma : [a,b] \to \Omega$
$\gamma \in C^1$ a tratti

$$\int_\gamma A = f(\gamma(b)) - f(\gamma(a)) \quad \text{(potenziale di } A\text{)}$$

$\Rightarrow$ se γ è chiusa $(\gamma(b) = \gamma(a))$ si ha $\int_\gamma A = 0$

Conseguenza Se $A \in C^0$ integrabile $\Rightarrow \int_\gamma A = 0$ $\forall$ γ chiusa in Ω.

$\gamma(t)$ $\quad \gamma : [a,b] \to \Omega$

$\sigma : [0,1] \to \Omega$ equivalente equiversa (uguale partenza e arrivo)

$\tilde{\sigma} : [0,1] \to \Omega$ " antiversa (partenza e arrivo di γ scambiati)

$$\sigma(s) = \gamma\big(a + s(b-a)\big) \qquad s \in [0,1]$$

$$\rho(s) = a + s(b-a) \leftarrow$$

$$\dot\rho(s) = b - a > 0$$

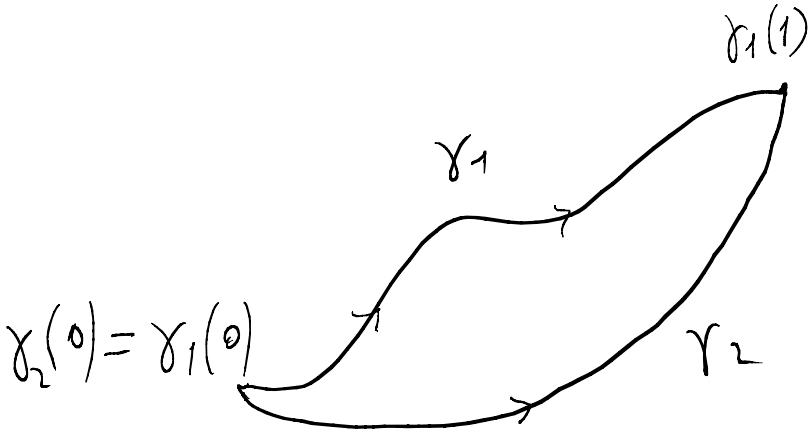
σ e γ sono equivalenti equiverse.

$$\tilde\sigma(s) = \gamma\big(b + s(a-b)\big) \qquad s \in [0,1]$$

equivalenti antiverse.

$$\tilde\rho(s) = b + s(a-b)$$

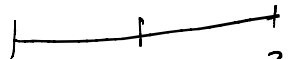
$$\dot{\tilde\rho}(s) = a - b < 0$$



$$\gamma_1, \gamma_2 : [0,1] \to \mathbb{R}^n$$

$$\boxed{\gamma_1(0) = \gamma_2(0) \qquad \gamma_1(1) = \gamma_2(1)}$$

$$\sigma(t) = \begin{cases} \gamma_1(t) & t \in [0,1] \\ \gamma_2(2-t) & t \in [1,2] \end{cases}$$



$2 - t \in [0,1]$ se $t \in [1,2]$

$$\int_\sigma A = \int_0^2 A(\sigma(t))\,\dot\sigma(t)\,dt =$$

$$= \int_0^1 A(\sigma(t))\,\dot\sigma(t)\,dt + \int_1^2 A(\sigma(t))\,\dot\sigma(t)\,dt =$$

$$= \int_0^1 A(\gamma_1(t))\,\dot\gamma_1(t)\,dt + \int_1^2 A(\gamma_2(2-t))\,\dot\gamma_2(2-t)\,(-1)\,dt$$

$$\int_{\gamma_1} A \qquad - \int_{\gamma_2} A$$

C.S. perché $A \in C^0$ sia integrabile è che $\int_\gamma A$ dipenda solo dagli estremi (e dal loro ordine) e non dal cammino $\gamma([a,b])$

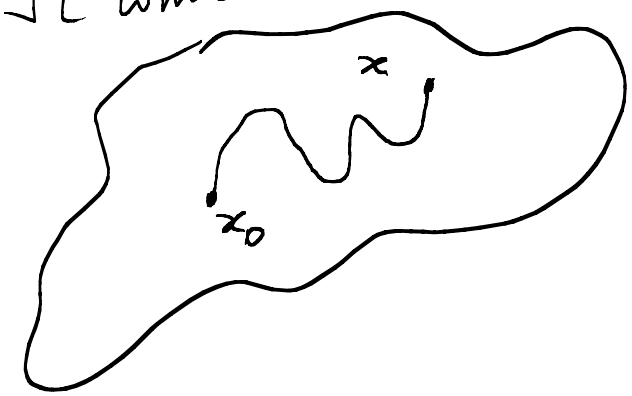
$$\left(\int_a^x f(t)\,dt \right)' = f(x)$$

DIM Fissiamo $x_0 \in \Omega$

γ curva qualunque che "parte" da x_0 e "arriva" a x $\quad \gamma : [a,b] \to \Omega \quad \gamma(a) = x_0 \quad \gamma(b) = x$

$$\to \quad f(x) = \int_\gamma A$$

Ω connesso



f è ben definita perché $\int_\gamma A$ non dipende dalla γ ma solo da $\gamma(a) = x_0$ e $\gamma(b) = x$

PROVIAMO CHE $\nabla f \equiv A$ su Ω $\iff$ $f_{x_i} \equiv A_i \quad \forall i = 1 \dots n$

$$\frac{f(x+he_i)-f(x)}{h} =$$

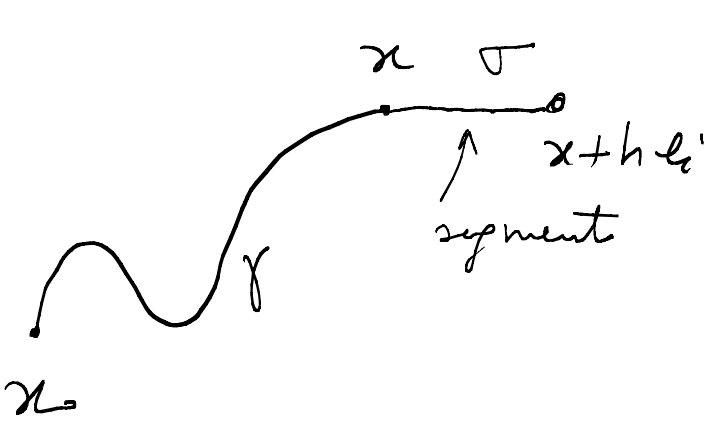
$$= \frac{1}{h}\left[\int_{\gamma\oplus\sigma} A - \int_{\gamma} A\right] =$$

($f(x)$)

$$= \frac{1}{h}\int_{\sigma} A = \frac{1}{h}\int_0^h A(x+te_i)\,e_i\,dt =$$

($A(\sigma(t))\,\dot\sigma$)

$$= \frac{1}{h}\int_0^h \underbrace{A_i(x+te_i)\,dt}_{\text{medio integrale}} =$$



$\gamma : [0,1] \to \Omega$

$\gamma(0) = x_0 \quad \gamma(1) = x$

$\sigma(t) = x + te_i \quad t \in [0,h]$

$$\int_{\gamma\oplus\sigma} = \int_{\gamma} A + \int_{\sigma} A$$

$$\int_a^{x+h} - \int_a^x = \int_a^x + \int_x^{x+h} - \int_a^x$$

$$= A_i(x+\xi e_i)$$

th. media integrale

ξ è compreso fra 0 e h

$\Downarrow$

$h \to 0 \Rightarrow \xi \to 0$

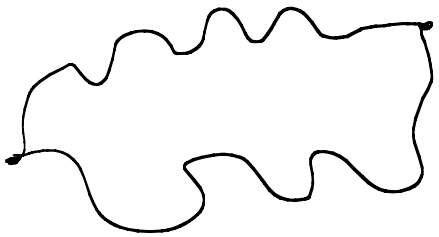
Th. confronto a tre (Carabinieri)

$$\lim_{h\to 0} \frac{1}{h}\left[f(x+he_i)-f(x)\right] = \lim_{h\to 0} A_i(\underbrace{x+\xi e_i}_{\xrightarrow{h\to 0} x}) = A_i(x)$$

$\forall i = 1..n$

$\alpha \equiv y\,dx + x\,dy$ forma

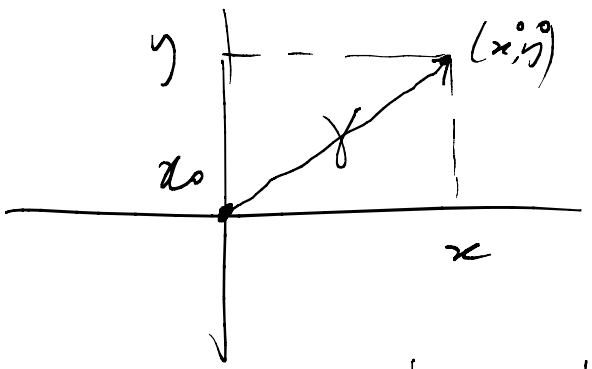
$A \equiv (y, x)$ campo



$$\boxed{\frac{\partial}{\partial y}(y) = 1 = \frac{\partial}{\partial x}(x)}$$

NULLA OSTA A CHE SIA INTEGR.

$A : \mathbb{R}^2 \to \mathbb{R}^2 \qquad x_0 = (0,0)$



$$A(x,y) = \begin{pmatrix} y \\ x \end{pmatrix}$$

$$\gamma(t) = t\begin{pmatrix} x^0 \\ y^0 \end{pmatrix} \qquad \begin{cases} x(t) = t x_0 \\ y(t) = t y_0 \end{cases} \qquad t \in [0,1]$$

$$\gamma(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \gamma(1) = \begin{pmatrix} x^0 \\ y^0 \end{pmatrix}$$

$$\dot{\gamma}(t) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$f(x^0;y^0) = \int_\gamma A =$$

$$= \int_0^1 A(\gamma(t))\,\dot{\gamma}(t)\,dt$$

$$= \int_0^1 (t y_0 x_0 + t x_0 y_0)\,dt =$$

$$= 2 x_0 y_0 \int_0^1 t\,dt = x_0 y_0 \, t^2 \Big|_0^1 =$$

$$= x_0 y_0$$

Il valore di (x_0, y_0) è ottenuto

Tutte le primitive saranno

$$xy + c$$

perché il dominio è connesso

(una volta contenti)

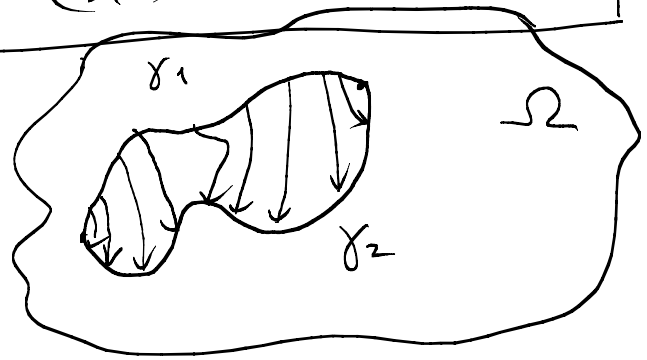
$$f(x,y) = xy$$

$$f_x(x,y) = y$$

$$f_y(x,y) = x$$

$$\nabla f \equiv \begin{pmatrix} y \\ x \end{pmatrix} = A$$

OMOTOPÌA (o DEFORMAZIONE)



$\gamma_1, \gamma_2 : [0,1] \longrightarrow \Omega$ continue (C^0)

$\gamma_1(0) = \gamma_2(0)$ e $\gamma_1(1) = \gamma_2(1)$

si dicono OMÒTOPE in Ω se esiste $h : [0,1] \times [0,1] \longrightarrow \Omega$

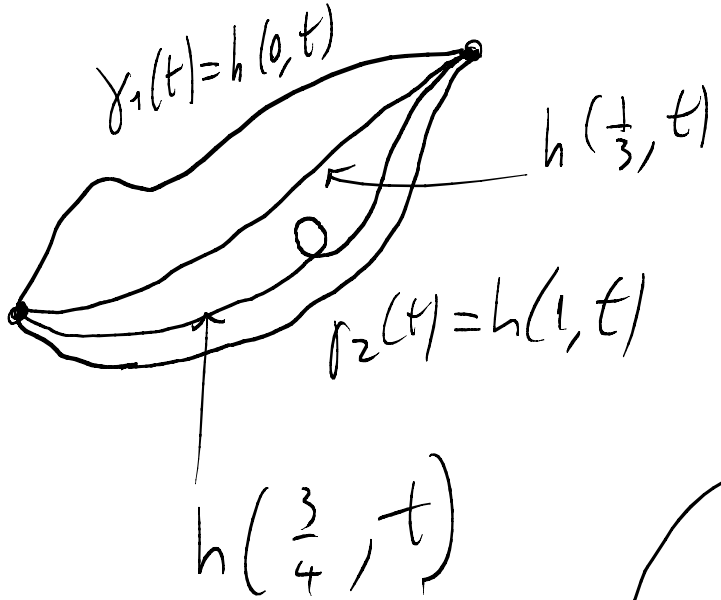
detta OMOTOPIA, tale che

1) h è continua su $[0,1]\times[0,1]$

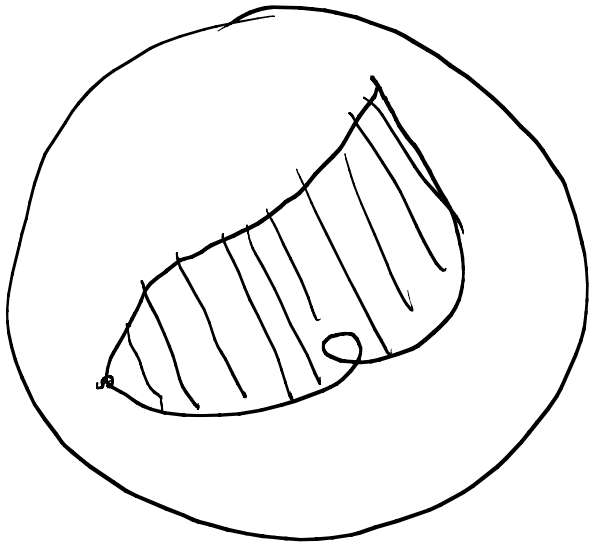
2) $h(0,t)=\gamma_1(t)$ $\quad h(1,t)=\gamma_2(t)$ $\quad \forall t\in[0,1]$

3) $h(\lambda,0)=\gamma_1(0)$

$h(\lambda,1)=\gamma_1(1)$



"γ_λ"$(t)=h(\lambda,t)$



ANALOGAMENTE $\gamma_1 : [0,1] \to \Omega$ CHIUSE
$\gamma_2 : [0,1] \to \Omega$

si dicono OMOTOPE IN Ω se

$\exists h : [0,1] \times [0,1] \to \Omega$, continua

tale che $h(0,t) = \gamma_1(t)$ e $h(1,t) = \gamma_2(t) \quad \forall t \in [0,1]$

Th. di INVARIANZA OMOTOPICA dell'integrale dei CAMPI IRROTAZIONALI

Sia $A \in C^1(\Omega)$ irrotazionale, cioè

$$(A_i)_{x_j} = (A_j)_{x_i} \quad \forall i,j = 1 \dots n$$

Siano γ_1, γ_2 due curve omotope (chiuse e con estremi comuni) in Ω.

Allora $\int_{\gamma_1} A = \int_{\gamma_2} A$

(L'integrale è invariante per omotopia)