Note Title 4/20/2021

DERIVATE SECONDE

$f : \Omega \to \mathbb{R}$

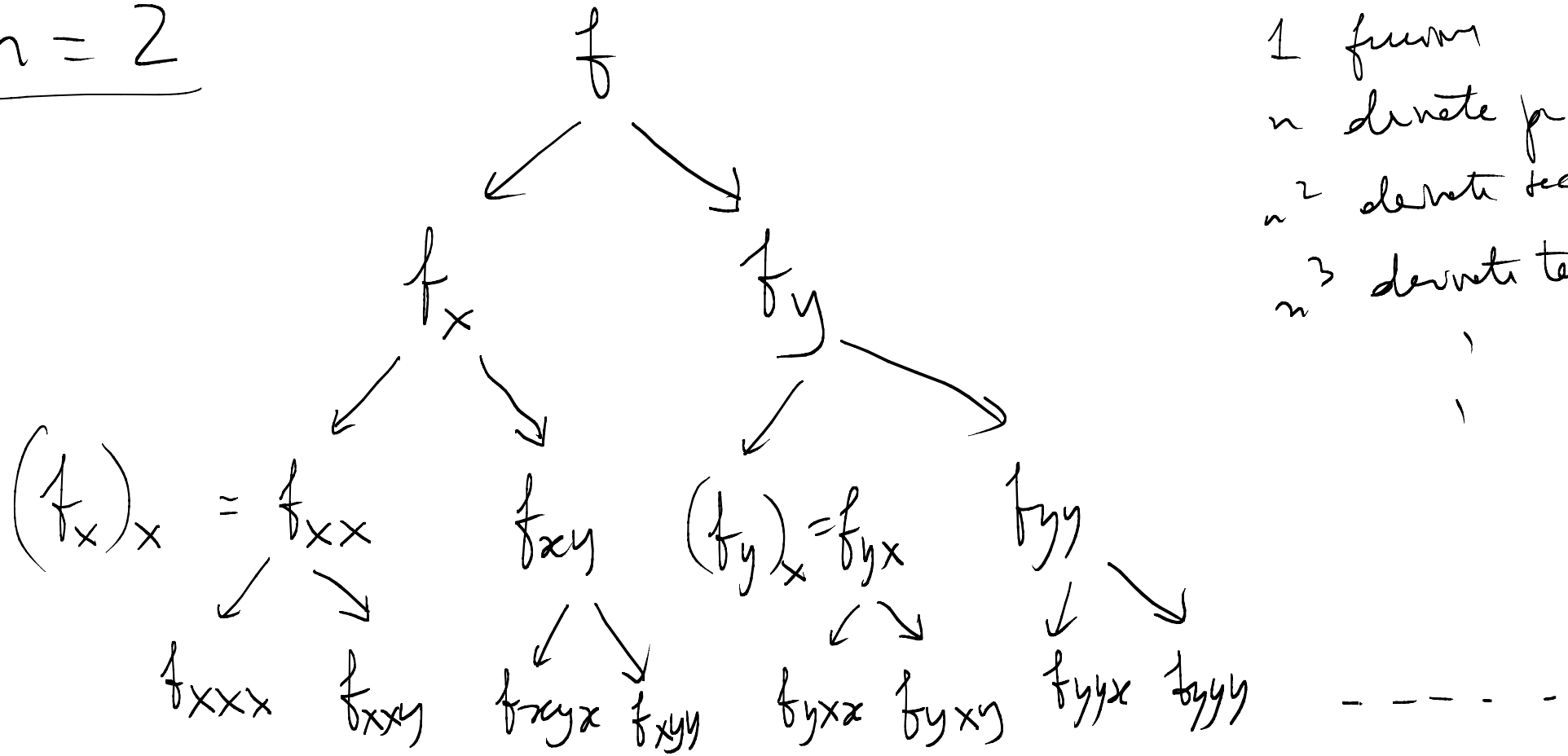
$\Omega \subseteq \mathbb{R}^n$

$f_{x_1 x_2} \equiv (f_{x_1})_{x_2}$ ← si esegue prima la derivata più vicina al simbolo f

$\dfrac{\partial^2}{\partial x_2 \partial x_1} f =$ ↑ ↙

$\dfrac{\partial^2}{\partial x \partial y}(x^2 y^2) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial}{\partial y} x^2 y^2\right) = \dfrac{\partial}{\partial x}(2x^2 y) = 4xy$

$n = 2$



1 funzione

n derivate prime

n^2 derivate seconde

n^3 derivate terze

$n = 3 \quad x, y, z$

$f,\ f_x, f_y, f_z,\ f_{xx}, f_{xy}, f_{xz}, f_{yx}, f_{yy}, f_{yz}, f_{zx}, f_{zy}, f_{zz}$ - - -

TEOREMA DI CLAIRAUT - SCHWARZ

(clerò — schwarz)

DEF $f \in C^k(\Omega)$ se f è dotata di tutte le derivate di ordine k, e sono continue in Ω

$$f \in C^2(\Omega) \iff f, f_{x_1}, \dots, f_{x_n}, f_{x_1x_1}, f_{x_1x_2}, \dots, f_{x_1x_n}, \dots$$
$$\dots\dots f_{x_nx_n} \in C^0(\Omega)$$

Th $f \in C^2(\Omega)$ allora $f_{x_ix_j} \equiv f_{x_jx_i}$

$$f(x,y) = x^3 y^2 \in C^0(\mathbb{R}^2)$$

$$f_x = 3x^2 y^2 \in C^0(\mathbb{R}^2)$$

$$\rightarrow f_y = 2x^3 y \in \text{"}$$

f_{xx}

$f_{xy} = f_{yx}$

$$f_{xy} = 6x^2 y$$

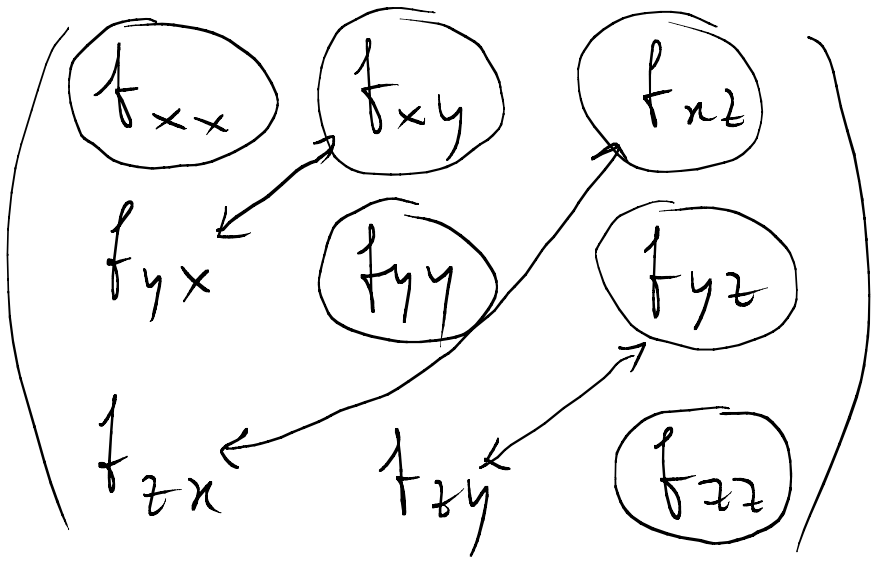
"mixed derivatives equal"

$$(f_y)_x = 6x^2 y \quad \text{(Clairaut-Schwarz)}$$

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} \leftarrow \text{Hessiana}$$

$f_{xy} = f_{yx}$ $\Rightarrow$ SIMMETRICA

$n=3$



A (autoagg. $A_{ij} = \overline{A_{ji}}$ le reali $A_{ij} = A_{ji}$ simmetriche)

Hessiana

simmetrica

Se $f \in C^2(\Omega)$ ogni derivata seconda NON dipende dall'ordine con cui viene derivata

$\Rightarrow f_{xyx} = f_{xxy}$ Se $f \in C^3 \Rightarrow f_x \in C^2$ $\xrightarrow{\text{Schwarz}}$

$f \in C^N(\Omega)$, $\Omega \subseteq \mathbb{R}^n$, Ω aperto, $x_0, x \in \Omega$

$$f(x) = \sum_{k=0}^{N} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k + R_N(x-x_0) \quad (*)$$

$w = x - x_0$

PEANO $R_N(x-x_0) = o(|x-x_0|^N)$

$f \in C^{N+1}$

LAGRANGE $R_N(x-x_0) = \frac{1}{(N+1)!} f^{(N+1)}(\xi)(x-x_0)^{N+1}$

$\xi \in [x_0, x]$

k multiindice

$k = (k_1\ k_2 \dots k_n)$ $\quad k! = k_1!\, k_2! \dots k_n!$

$|k| = k_1 + k_2 + \dots + k_n$ $\quad f^{(k)} = \dfrac{\partial^{|k|}}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_n^{k_n}} f$

$w^k = w_1^{k_1} w_2^{k_2} \dots w_n^{k_n}$ $\quad w_i^0 = 1 \quad 0! = 1$

La formula di Taylor (Gregory) vale

in più variabili interpretando k come multindice

$$f(x_0 + w) = \sum_{|k|=0}^{N} \frac{1}{k!} f^{(k)}(x_0)\, w^k + R_N(w) =$$

$$= \sum_{\substack{k_1,\dots k_n \geq 0 \\ \sum_1^n k_i \leq N}} \frac{1}{k_1!\, k_2! \cdots k_n!} \frac{\partial^{(\sum k_i)}}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_n^{k_n}} f(x_0)\, w_1^{k_1} w_2^{k_2} \cdots w_n^{k_n} +$$

$$+ R_N(w) =$$

POLINOMIO DI TAYLOR

RESTO

$$= \sum_{M=0}^{N} \sum_{\substack{k_1, k_2 \cdots k_n \geq 0 \\ \sum k_i = M}} \frac{1}{k_1! \cdots k_n!} f_{x_1^{k_1} \cdots x_n^{k_n}}(x_0)\, w_1^{k_1} \cdots w_n^{k_n} + \text{R-esto}$$

somma dei termini di grado $M = \sum_{1}^{n} k_i$

n, M fissati

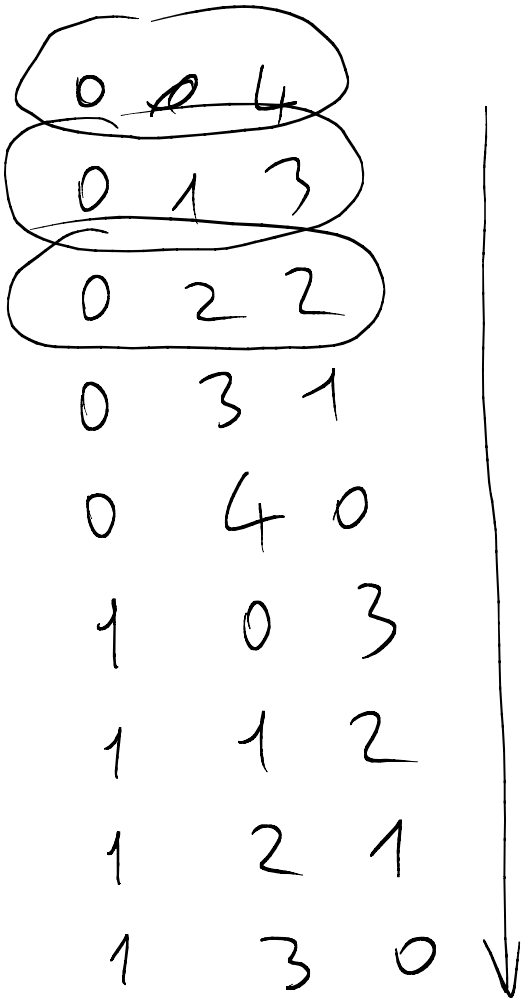
$n = 3$ $\quad$ $M = 3$

k_1	k_2	k_3
0	0	3
0	1	2
0	2	1
0	3	0
1	0	2
1	1	1
1	2	0

$k_3 = 3 - k_1 - k_2$

k_1	k_2	k_3
2	0	1
2	1	0
3	0	0

$n = 3 \qquad M = 4$



$f(x,y) = \sin(xy)$

$x_0 = (0,0)$

$w = \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$

$(x,y) = (0,0) + \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$

$x_0 \quad w$

$f(0,0) = 0$

$$f(x,y) = f(0,0) + \underbrace{f_x(0,0)(x-x_0) + f_y(0,0)(y-y_0)}_{df((0,0);w)} +$$

$$+ \frac{1}{0!2!} f_{yy}(0,0)(y-y_0)^2 + \frac{1}{1!1!} f_{xy}(0,0)(x-x_0)(y-y_0) +$$

$$+ \frac{1}{2!0!} f_{xx}(0,0)(x-x_0)^2$$

(0 2)

$w_1^0 \, w_2^2$

$M = k_x + k_y = 2$

0	2
1	1
2	0

$\sin(xy)$ $\qquad f_{xx} = y(-\sin(xy)\,y) = -y^2 \sin xy$

$$\begin{cases} f_x = \cos(xy)\,y & f_{xy} = -\sin(xy)\,xy + \cos(xy)\cdot 1 \\ f_y = \cos(xy)\,x & f_{yy} = -\sin(xy)\,x^2 \end{cases}$$

$$f_{xx}(0,0) = f_{yy}(0,0) = 0 \qquad f_{xy}(0,0) = 1$$

$$\sin(xy) = 0 + 0 + 1 \cdot xy + R_2(x,y)$$

$$\boxed{\sin(xy) = \underbrace{xy}_{f_{xy}(0,0)} + R_2(xy)} \qquad \lim_{(x,y)\to(0,0)} \frac{R_2(x,y)}{|(x,y)|^2} = 0$$

$f \in C^2[a,b]$, $x_0 \in]a,b[$, $f'(x_0) = 0$ (Fermat)

$$f(x_0+w) = f(x_0) + f'(x_0)w + \frac{1}{2!} f''(x_0) w^2 + o(w^2)$$

$$\underbrace{f(\underbrace{x_0+w}_{x}) - f(x_0)}_{\text{il segno determina se } x_0 \text{ è di max, min o altro...}} = w^2 \left[\frac{f''(x_0)}{2} + \frac{o(w^2)}{w^2} \right]$$

il segno è facile da studiare

$w^2 \geq 0$

$$[\ ] \xrightarrow[w \to 0]{} \frac{f''(x_0)}{2}$$

perché $\frac{o(w^2)}{w^2} \xrightarrow[w \to 0]{} 0$

Per la permanenza del segno $[\ ]$ ha lo stesso segno di $f''(x_0)$ dove

$$\begin{cases} f''(x_0) > 0 & x_0 \text{ è di minimo} \\ f''(x_0) < 0 & x_0 \text{ è di max} \end{cases}$$

$$\boxed{f''(x_0) = 0} \quad ? \; ? \; ? \; ? \quad \begin{matrix} x^3 \\ x^4 \end{matrix}$$

$$f'(x_0) = \underline{f''(x_0)} = \dots = \underline{f^{(k-1)}(x_0)} = 0 \qquad \boxed{f^{(k)}(x_0) \neq 0}$$

$$f(x_0 + w) = f(x_0) + \frac{1}{k!} f^{(k)}(x_0)\, w^k + o(w^k) \iff$$

$$f(x_0 + w) - f(x_0) = \underline{\underline{w^k}} \underbrace{\left[\frac{f^{(k)}(x_0)}{k!} + \frac{o(w^k)}{w^k} \right]}_{\text{ha lo stesso segno, in un intorno di } f^{(k)}(x_0)}$$

$f \in C^2 \qquad f'(x_0) \equiv \nabla f(x_0) = 0$

$$f(x_0+w) - f(x_0) = \cancel{\nabla f(x_0) w} + \frac{1}{2!} \sum f_{x_i x_j}(x_0) w_i w_j + o(|w|^2)$$

forma quadratica