Note Title 3/30/2021

$$\lim_{0} \frac{1}{x^2+2y^2-2xy}$$ $\frac{1}{f}$ ove $f \to 0$ per $x \to 0$

$$\lambda |x|^2 \le \sum_{ij} \alpha_{ij} x_i x_j \le \Lambda |x|^2$$

AL_7.13

$x^2 - 2xy + 2y^2 = (x-y)^2 + y^2 \ge 0$ e si annulla se e solo se

$$\begin{cases} x-y=0 \\ y=0 \end{cases} \iff x=y=0$$

DEFINITA POSITIVA

$$\lim_{0} \frac{1}{x^2+2y^2-2xy} = +\infty$$

$\lim_{\infty} \sum \alpha_{ij} x_i x_j$

$$\lambda |x|^2 \leq \sum \alpha_{ij} x_i x_j \leq \Lambda |x|^2$$

Th. confronto

Def. >0 $\lambda > 0 \Rightarrow \sum \alpha_{ij} x_i x_j$ diverge $+\infty$

Def <0 $\Lambda < 0 \Rightarrow$ " " " $-\infty$

in tutti gli altri casi $\sum \alpha_{ij} x_i x_j$ oscilla

$\lambda = 0$ (semidefinito >0) sull'autospazio di λ H è cost. nulla

$H(x,y) = x^2$ in $\mathbb{R}^2$ $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ diag.

semidef. >0
$\lambda = 0$ $\Lambda = 1$

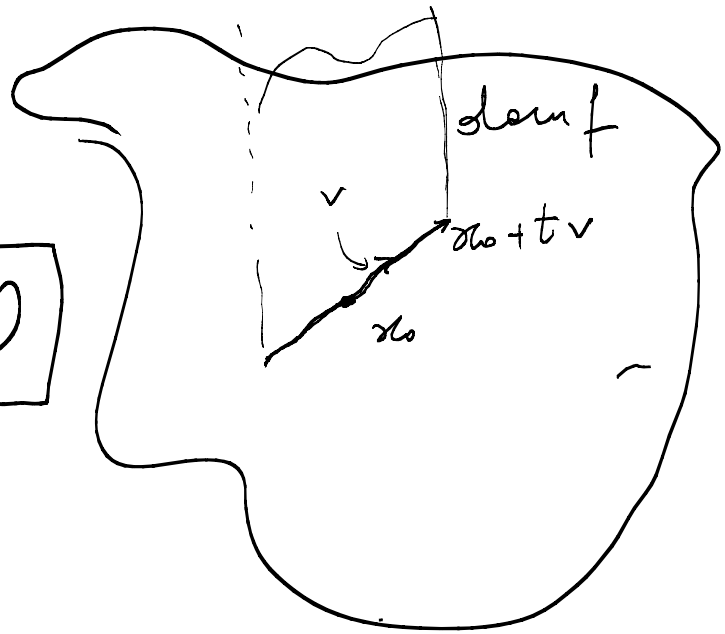
$H(0,y) \equiv 0$

$H(x,0) = x^2 \to +\infty$

AN_2.1 → AN_2.6

DERIVATA DIREZIONALE

$V \neq 0$



$$h(t) = f(x_0 + tv)$$

$$f_v(x_0) = \lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t}$$

se il limite esiste finito

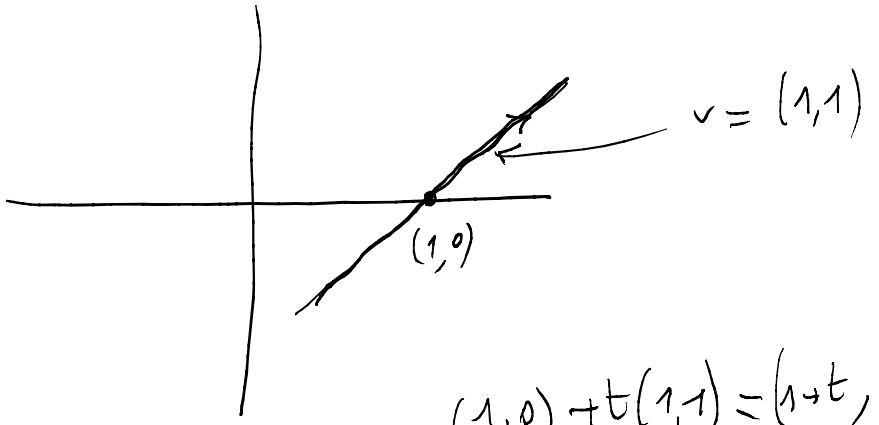
OSS $$f_v(x_0) = \lim_{t \to 0} \frac{h(t) - h(0)}{t} = h'(0)$$

SIMBOLI ALTERNATIVI

$f_v(x_0)$ $\partial_v f(x_0)$ $\frac{\partial f}{\partial v}(x_0)$

SINONIMI

$$\frac{\partial (x^2 y)}{\partial v}\underbrace{(1,0)}_{x_0} \qquad v = (1,1)$$



$$f(x,y) = x^2 y$$

$$\lim_{t\to 0} \frac{f(\underbrace{(1,0)}_{x_0} + t\underbrace{(1,1)}_{v}) - f(1,0)}{t} = \lim_{t\to 0} \frac{f(\overbrace{(1,0)+t(1,1)}^{(1+t,t)}) - f(1,0)}{t} =$$

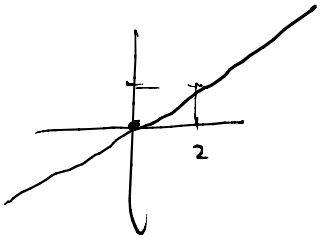
$$= \lim_{t\to 0} \frac{(1+t)^2 t - 0}{t} = \lim_{t\to 0} (1+t)^2 = 1$$

$$h(t) = f((1,0)+t(1,1)) = f(1+t,t) = (1+t)^2 t$$

$$f_{(1,1)}(1,0) = h'(0)$$

$$h'(0) = (1+t)^2 + 2(1+t)t \Big|_{t=0} = 1$$

$$f(x,y) = \begin{cases} 1 & (0,0) \\ \dfrac{\sin(x^2+y^2)}{x^2+y^2} & (x,y) \neq (0,0) \end{cases}$$



$(0,0) + t(2,1) = (2t, t)$

$$f_{(2,1)}(0,0) = \lim_{t\to 0} \frac{1}{t}\left[\frac{\sin(4t^2+t^2)}{4t^2+t^2} - 1 \right] = \lim_{t\to 0} \frac{\sin(5t^2) - 5t^2}{5t^3} =$$

($f(2t,t)$, $f(0,0)$)

$\sin w - w \sim O(w^3)$

$w - \frac{w^3}{3!} + o(w^3)$

$$= -\frac{1}{6} \leftarrow \frac{\sin(5t^2) - 5t^2}{125t^6} \cdot \frac{125t^6}{5t^3} \to 0$$

$f(x,y) = |xy|$

$\frac{\partial f}{\partial (1,0)}(0,0) = 0$

$(0,0) + t(1,0) = (t,0)$

$h(t) = f(t,0) \equiv 0$

$\Rightarrow h'(0) = 0 \Rightarrow$

$\frac{\partial f}{\partial (1,1)}(0,0)$

$h(t) = f(t,t) = |t^2| = t^2$

$\underline{h'(0)} = 2t\Big|_{t=0} = 0$

$g(x,y) = |xy|^{1/2}$

$g_{(1,1)}(0,0) =$

$h(t) = g(t,t) = \sqrt{|t^2|} = |t|$

NON È DERIVABILE IN $t=0$

DEF

$$\frac{\partial f}{\partial e_i}(x_0) \equiv \frac{\partial f}{\partial x_i}(x_0)$$

DERIVATA PARZIALE di f rispetto a x_i

$f(x,y)$ $\quad (x_0, y_0)$

$v = e_1 = (1,0)$

$tv = (t,0)$

$\frac{\partial f}{\partial x_i}$, f_{x_i} , $\partial_{x_i} f$

$$\frac{\partial f}{\partial x}(x_0,y_0) = \lim_{t\to 0} \frac{1}{t}\Big[\underbrace{f(x_0+t, y_0+0t)}_{(x_0,y_0)+te_1} - f(x_0,y_0)\Big] =$$

$$\lim_{0} \frac{f(x_0+t, y_0) - f(x_0,y_0)}{t}$$

è la derivata della funzione ottenuta da f bloccando tutte le variabili salvo $\underline{x}$

La notazione

$$f(x_1, \dots, x_n)$$

$$g(t) = f(x_1^0 + t, x_2^0, \dots, x_n^0) \leftarrow$$

allora $\dfrac{\partial f}{\partial x_1}(x_1^0, \dots, x_n^0) = g'(0)$

Si devono supporre costanti tutte le variabili non coinvolte nella derivazione

$$\frac{\partial}{\partial x}(x^2 y) = 2xy \qquad \frac{\partial}{\partial y}(x^2 y) = x^2$$

(cost.)

$$\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right) = 2x \qquad \frac{\partial}{\partial z}\left(x^2+y^2+z^2\right) = 2z$$

costanti in x — costanti in z — $f(x,y,z)$

$$f : \Omega \longrightarrow \mathbb{R}$$

1) $x_0 \in \mathring{\Omega}$ (x_0 interno ad Ω)

2) x_0 sia di massimo o di minimo locale

3) in x_0 f abbia una derivata direzionale $f_v(x_0)$

$$\Longrightarrow \quad f_v(x_0) = 0$$

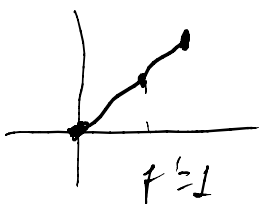
x_0 è di minimo locale per f

se $\exists \delta > 0$:

$$f(x_0) \leq f(x) \quad \forall x \in \mathrm{dom} f \cap B_\delta(x_0)$$

x_0 max locale se ....

$$f(x_0) \geq f(x)$$



$f(x) = x$ su $[0,1]$

$f'(0) = f'(1) = 1$

$x = 0$ minimo globale

$\Rightarrow$ locale

$f'(0)$ non esiste

DIM Perché $x_0 \in \mathring{\Omega}$ (x0 interno ad Ω)

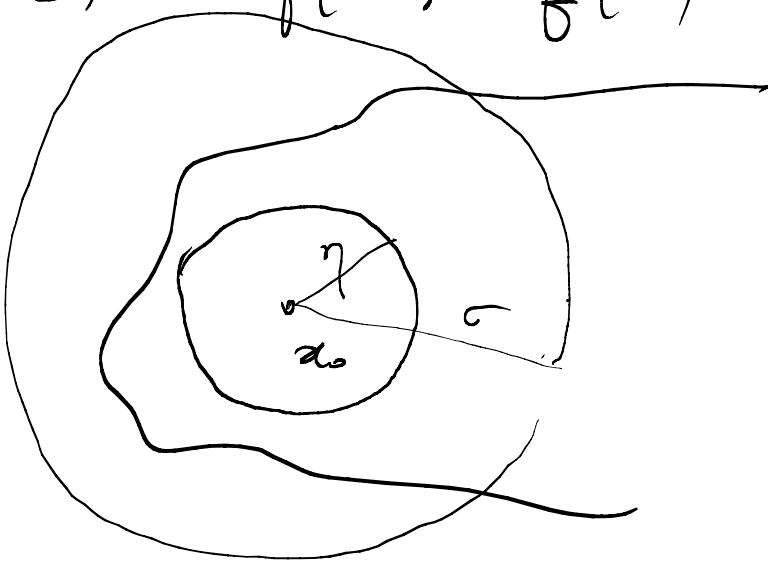
$$\exists \eta > 0 : B_\eta(x_0) \subseteq \Omega$$

Perché x_0 è di minimo locale per f

$$\exists \sigma > 0 : \forall x \in \mathrm{dom} f \cap B_\sigma(x_0) \quad f(x_0) \leq f(x)$$

$$\delta = \min\{\eta, \sigma\}$$

$$\rightarrow \boxed{f(x_0) \leq f(x) \quad \forall x \in B_\delta(x_0)}$$



Per ipotesi 3) $\exists f_v(x_0)$

$$|x_0 + tv - x_0| = |t||v|$$

e quindi $x_0 + tv \in B_\delta(x_0)$ se $|t||v| < \delta \iff |t| < \frac{\delta}{|v|}$

$$\left]-\frac{\delta}{|v|}, \frac{\delta}{|v|}\right[$$

$h(t) = f(x_0 + tv)$ $\quad (h'(0) = f_v(x_0))$

h è definita su $\left]-\frac{\delta}{|v|}, \frac{\delta}{|v|}\right[$

0 è interno a $\left]-\delta/|v|, \delta/|v|\right[$

h ha un minimo locale in 0 perché $h(0) = f(x_0) \leq f(x_0 + tv) = h(t)$

Per il teorema di Fermat $h'(0) = 0$ ∎

$f(x,y) = |xy|^{1/2}$ ha solo 2 derivate direzionali: quelle parziali $\begin{cases} f_x(0,0) = 0 \\ f_y(0,0) = 0 \end{cases}$

in ogni altra direzione non è derivabile

$(0,0)$ è di minimo globale

NON È VERO CHE LA DERIVABILITÀ IMPLICHI LA CONTINUITÀ

$$f(x,y) = \begin{cases} 0 & (0,0) \\ \left(\dfrac{x^2 y}{x^4+y^2}\right)^2 & (x,y) \neq (0,0) \end{cases}$$

1) f ha <u>tutte</u> le derivate direzionali in $(0,0)$ (e sono 0)

2) ma è discontinua in $(0,0)$.

1) $v = (\alpha, \beta)$

$(0,0) + t(\alpha,\beta) = (\alpha t, \beta t)$

$$f_v(0,0) = \lim_{t\to 0} \frac{1}{t}\left[\left(\frac{\alpha^2 t^2 \beta t}{\alpha^4 t^4 + \beta^2 t^2}\right)^2 - 0\right] =$$

$$= \frac{1}{t} \frac{\alpha^4 \beta^2 t^6}{t^4(\alpha^4 t^2 + \beta^2)^2} \overset{t\to 0}{=} \begin{cases} \to 0 & \beta \neq 0 \\ \equiv 0 & \beta = 0 \end{cases}$$

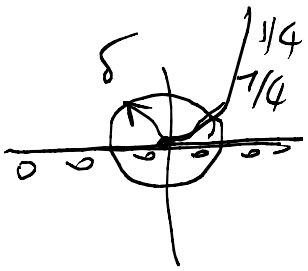
$t^2(\alpha^4 t^2 + \beta^2)$

2) $y = kx^2$

$$f(x, kx^2) \underset{x \neq 0}{=} \left(\frac{x^2 \cdot kx^2}{x^4 + k^2 x^4} \right)^2 =$$

(x^2 : y ; $x^4 + y^2$)

$$= \left(\frac{x^4 k}{x^4 (1+k^2)} \right)^2 = \frac{k^2}{(1+k^2)^2}$$



e quindi f sulla vicina a 0

per $k=0 \Rightarrow$ circa $f = 0$ per $k=1$ $y = x^2$

f vale $\frac{1}{4}$