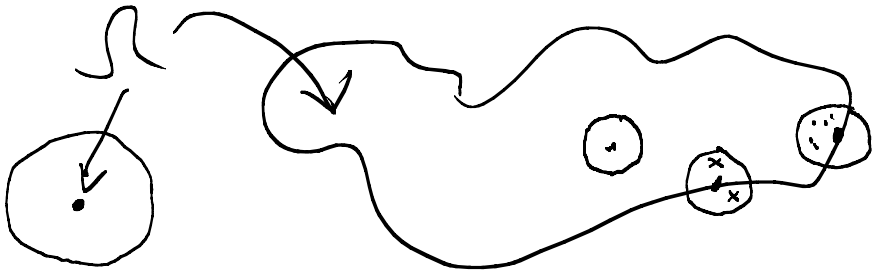
Note Title 3/9/2021

x_0 è di accum. per Ω se $\forall \varepsilon > 0 \ \exists x \in \Omega \ |x - x_0| < \varepsilon \quad x \neq x_0$

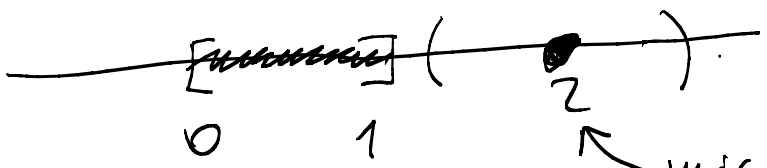


x_0 è di frontiera se

$\forall \varepsilon > 0 \ \exists x \in \Omega, y \notin \Omega$

: $|x - x_0| < \varepsilon \quad |y - x_0| < \varepsilon$

$[0,1] \cup \{2\} = \Omega$



x_0 si dice p. isolato di Ω

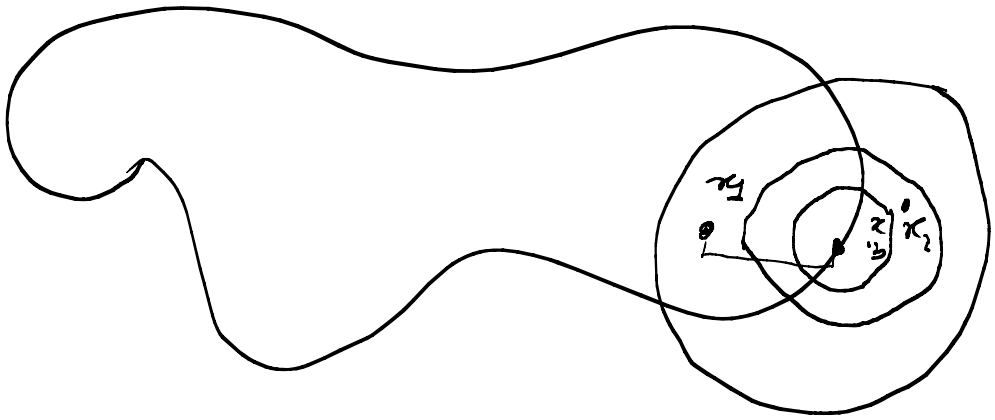
se 1) $x_0 \in \Omega$

2) $\exists \varepsilon > 0 : \ \Omega \cap B_\varepsilon(x_0) = \{x_0\}$

unico punto di Ω contenuto in $B_{\frac{1}{2}}(2)$

$\varepsilon < 1$

es $\boxed{\frac{1}{2}}$



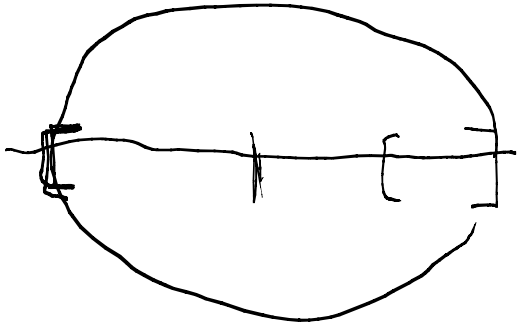
$x_n \in B_{\delta_n}(x_0)$

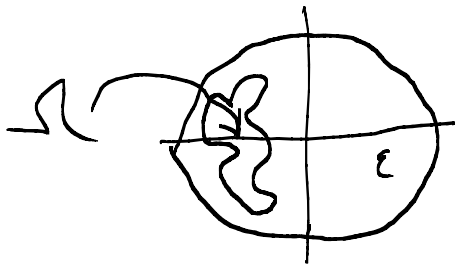
$\delta_n < \frac{1}{n}$

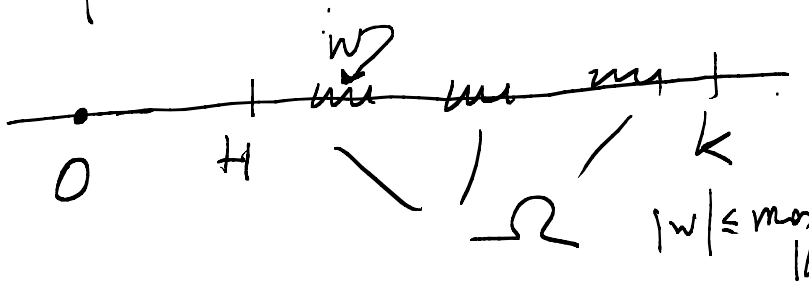
$\delta_n < \min\{|x_1 - x_0|, |x_2 - x_0| \dots, |x_{n-1} - x_0|, \frac{1}{n}\}$

Insieme limitato

Ω si dirà LIMITATO se $\exists \varepsilon > 0 : |w| < \varepsilon \ \forall w \in \Omega$







Funzione non limitata

$$\forall \varepsilon > 0 \quad \exists \bar{w} \in \Omega : |\bar{w}| \geq \varepsilon$$

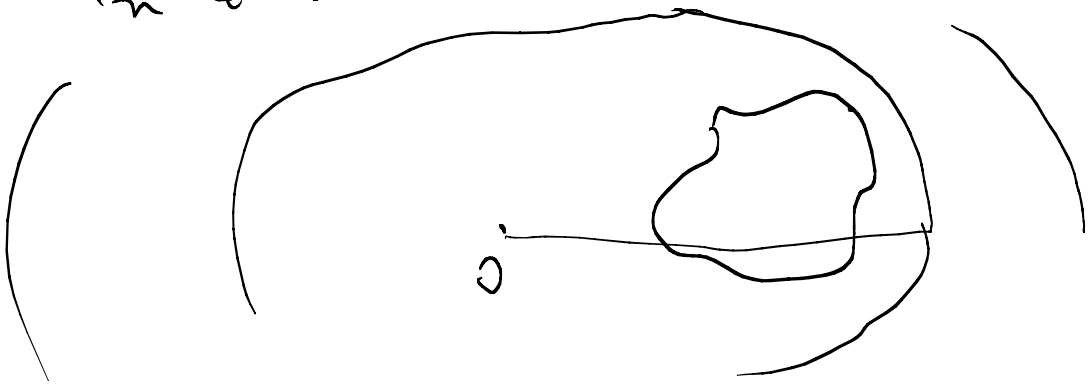
---

funzione (successione) limitata

$f : \Omega \to \mathbb{R}^p$ si dice limitata se l'immagine è limitata

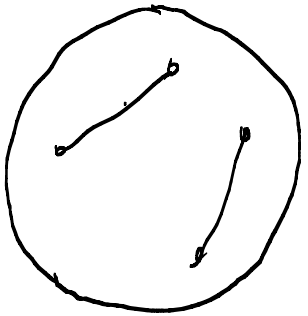
cioè $\exists \varepsilon > 0 : |f(x)| \leq \varepsilon \quad \forall x \in \Omega$

a_n è una succ. limitata se $\exists \varepsilon > 0 : |a_n| \leq \varepsilon \quad \forall n \in \mathbb{N}$

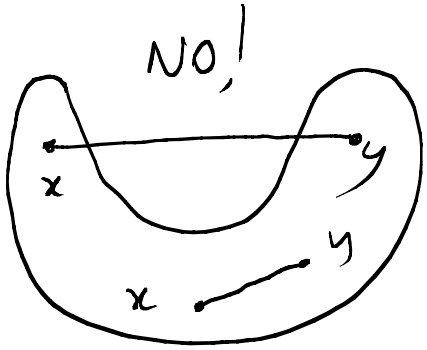


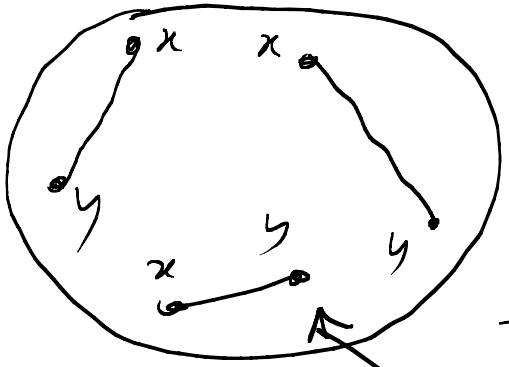
Ω INSIEME CONVESSO

$\forall x, y \in \Omega \quad \forall \lambda \in [0,1] \quad (1-\lambda)x + \lambda y \in \Omega$



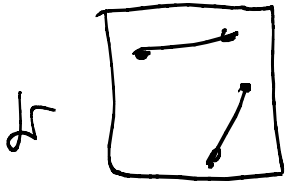
Ω



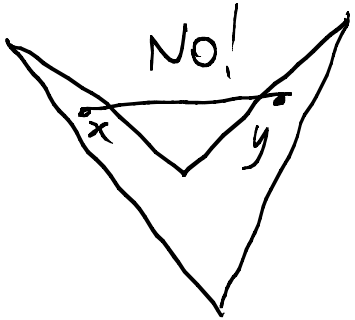


$\forall x, y$

Ω



Ω



Th

La sfera aperta $B_M(0)$ è convessa.

DIM

Siano $x, y \in B_M(0) \iff |x| < M \quad |y| < M$

E' vero che $(1-\lambda)x + \lambda y \in B(0,M) \quad \forall \lambda \in [0,1]$

$$|(1-\lambda)x + \lambda y| < M$$?

$$|(1-\lambda)x + \lambda y|^2 = (1-\lambda)^2|x|^2 + \underbrace{2\lambda(1-\lambda)}_{\geq 0}\, xy + \lambda^2|y|^2 \leq$$

(prodotto scalare)

$$\leq (1-\lambda)^2|x|^2 + 2\lambda(1-\lambda)|x||y| + \lambda^2|y|^2 <$$

Schwartz.

$$< (1-\lambda)^2 M^2 + 2\lambda(1-\lambda)M \cdot M + \lambda^2 M^2 =$$

$$= M^2\left[(1-\lambda)^2 + 2\lambda(1-\lambda) + \lambda^2\right] = M^2\left[\underbrace{(1-\lambda)+\lambda}_{=1}\right]^2 =$$

$$= M^2$$

Applicando la radice (tutti i termini sono ≥ 0)

$$|(1-\lambda)x + \lambda y| < M$$ ∎

$$\big||x| - |y|\big| \leq |x-y| \qquad \forall x, y \in \mathbb{R}^N$$

valore assoluto

DIM. $|x| = |x - y + y| \overset{\text{triang.}}{\leq} |x-y| + |y|$

quindi

$$|x| - |y| \leq |x-y| \longrightarrow \text{Teor}$$

$$|y| = |y - x + x| \leq |y-x| + |x| = |x-y| + |x| \Rightarrow |y| - |x| \leq |x-y|$$

Conseguenze

Th. La funzione $x \to |x|$ è continua da $\mathbb{R}^N$ in $\mathbb{R}$

Scelto $\varepsilon > 0$ ad arbitrio si pone $\delta = \varepsilon$

$$\big||x| - |y|\big| \le |x - y| < \varepsilon$$

IS $\forall \varepsilon > 0 \ \exists \delta > 0 : |x - y| < \delta \Rightarrow \big||x| - |y|\big| < \varepsilon$

Th. Se a_n è convergente allora è limitata.

Dim:

$$|a_n| = |a_n - L + L| \le |a_n - L| + |L|$$

Scelto $\varepsilon = 1 \quad \exists \nu : \forall n > \nu \quad |a_n - L| < 1$

ne segue che $\underline{|a_n| \le |L|+1 \quad \forall n > \nu}$

da cui

$$|a_n| \le \underbrace{\max\{|L|+1, |a_1|, |a_2|, \dots, |a_\nu|\}}_{\text{è che maggiora } |a_n| \ \forall n} \quad \forall n \in \mathbb{N}$$

Th. $a_n \to L \quad b_n \to M$ allora $a_n b_n \to LM$

prodotti scalari

DIM. Occorre provare che $\forall \varepsilon > 0 \ \exists \nu : \forall n > \nu \ |a_n b_n - LM| < \varepsilon$

$$|a_n b_n - LM| = |a_n b_n - a_n M + a_n M - LM| =$$

$$= |a_n (b_n - M) + M(a_n - L)| \le$$

triang. inf. valore assoluto

$$\leq |a_n(b_n - M)| + |M(a_n - L)| \leq$$

Schwarz

$$\leq |a_n||b_n - M| + |M||a_n - L| \leq$$

Poiché a_n è conv. allora è limitata e quindi $\exists k > 0$: $|a_n| \leq k \ \forall n \in \mathbb{N}$

$$= \underbrace{k|b_n - M|}_{< \frac{\varepsilon}{2}} + \underbrace{|M||a_n - L|}_{< \frac{\varepsilon}{2}} < \varepsilon$$

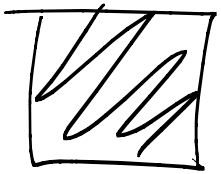
Per l'ipotesi $a_n \to L$ scelto $\frac{\varepsilon}{2|M|}$ $\exists \nu_1 : \forall n > \nu_1$

$$|a_n - L| < \frac{\varepsilon}{2|M|} \Rightarrow |M||a_n - L| < \frac{\varepsilon}{2}$$

Per l'ipotesi $\boxed{b_n \to M}$, scelto $\frac{\varepsilon}{2k}$ $\exists \nu_2 : n > \nu_2$

$$\boxed{|b_n - M| < \frac{\varepsilon}{2k}} \Rightarrow \underline{k|b_n - M| < \frac{\varepsilon}{2}}$$

$$\boxed{\forall n > \max\{\nu_1, \nu_2\} \equiv \nu}$$



Permanenza del segno

1) $f : \Omega \to \underline{\underline{\mathbb{R}}}$, $\Omega \subseteq \mathbb{R}^N$, continua in Ω (in ogni punto di Ω)

2) $f(x_0) > 0$ ←

TS $\exists \varepsilon > 0$: $f(x) > 0 \quad \forall x \in B_\varepsilon(x_0)$

DIM

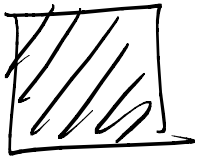
Per la continuità

$\forall \varepsilon > 0 \; \exists \delta > 0 : \forall x \in \text{dom} f \quad |x - x_0| < \delta \quad |f(x) - f(x_0)| < \varepsilon$

$|f(x) - f(x_0)| < \varepsilon \overset{\mathbb{R}}{\iff} f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon$

$\varepsilon = f(x_0) \underset{\text{cont.}}{\Longrightarrow} \exists \delta > 0 : \forall x \in \text{dom} f \quad |x - x_0| < \delta$

$0 \leq f(x_0) - \varepsilon < f(x)$



Se $f(x_0) < 0$ basta scegliere $\varepsilon = |f(x_0)| = -f(x_0)$

e usare la dis. $f(x) < f(x_0) + \varepsilon = 0$