

$X \oplus Y$ è lo spazio vettoriale $X + Y$, quando è
verificate le proprietà

(*) $x \in X, y \in Y, x + y = 0 \Rightarrow x = y = 0$

$u_1 \dots u_n$ base di X

$Y = \langle u_1, \dots, u_k \rangle \quad Z = \langle u_{k+1}, \dots, u_n \rangle$

T3.

$$Y \oplus Z$$

$$Y + Z = X$$

$\forall x \in X \quad \exists \alpha_i : \sum \alpha_i u_i = x$

$$x = \underbrace{\sum_{i=1}^k \alpha_i x_i}_{\in Y} + \underbrace{\sum_{j=k+1}^n \alpha_j x_j}_{\in Z}$$

\Rightarrow opere $\alpha \in X$
appartenente a
 $Y + Z$

$$X = Y + Z \leftarrow Y + Z \subseteq X$$

Le somme i dirette, fatti se $y + z = 0$ $y \in Y$ $z \in Z$

$$\boxed{\sum_{i=1}^k \alpha_i u_i + \sum_{j=k+1}^n \alpha_j u_j = 0} \Leftrightarrow \alpha_i = -\alpha_j$$

def. d'indip. di u_1, \dots, u_n

$$y = \sum_{i=1}^k \alpha_i u_i \quad z = \sum_{j=k+1}^n \alpha_j u_j$$

$$\boxed{\sum_0^n \alpha_i t^i = 0} \Rightarrow \alpha_i = 0 \quad \forall i = 0 \dots n$$

~~$t \neq 0$~~

$\boxed{p(t) = \sum_0^n \alpha_i t^i}$ $\boxed{0 = p(0) = \alpha_0}$

$$p(t) = t \sum_0^n \alpha_i t^{i-1}$$

|||
0

Se $t \neq 0$, per le leggi d'annullamento del prodotto

$$p_1(t) = \sum_1^n \alpha_i t^{i-1} = 0$$

Poiché tale polinomio è una funzione continua, ne segue

$\alpha_1 = p_1(0) = \lim_{t \rightarrow 0} p_1(t) = 0$

continuità

perciò $p_1(t) = 0 \quad \forall t \neq 0$

$\boxed{x_1 = 0}$

$\boxed{AL - 3.1}$

Z dim fatto X, Y sottoinsiemi di Z allora

$$\underbrace{\dim X+Y}_{\dim X \cap Y = k} + \underbrace{\dim X \cap Y}_k = \underbrace{\dim X}_{n} + \underbrace{\dim Y}_{m}$$

$$X \cap Y \subset X$$

$$\dim X \cap Y = k > 0$$

$$X \cap Y \subset Y$$

$$\dim X+Y = n+m-k$$

Sono w_1, \dots, w_k base di $X \cap Y$.

Per complemento, sono

$$\underline{w_1, \dots, w_k, u_{k+1}, \dots, u_n} \text{ base di } X$$

$$\underline{w_1, \dots, w_k, v_{k+1}, \dots, v_m} \text{ base di } Y$$

AL_3.5

Si presume che $w_1 \dots w_k, u_{k+1} \dots u_n, v_{k+1} \dots v_m$ sono

basi di $X+Y$ e dunque $\dim X+Y = k+(n-k)+(m-k) = n+m-k$

$$1) X+Y = \langle w_1 - w_k, u_{k+1} \dots u_n, v_{k+1} \dots v_m \rangle$$

$$\forall z \in X+Y \Rightarrow z = x+y = \underbrace{\sum_{i=1}^k \alpha_i w_i}_{\in X} + \underbrace{\sum_{j=k+1}^n \beta_j u_j}_{\in Y} + \underbrace{\sum_{l=1}^k \gamma_l v_l}_{\in Y} + \underbrace{\sum_{l=k+1}^m \delta_l v_l}_{\in Y}$$

$$2) \text{ I vettori sono indip.} \quad \text{Sia} \quad \underbrace{\sum_{i=1}^k \alpha_i w_i}_{w} + \underbrace{\sum_{j=k+1}^n \beta_j u_j}_{x} + \underbrace{\sum_{l=k+1}^m \gamma_l v_l}_{y} = 0$$

$$w+x+y=0 \Rightarrow \underbrace{w+x}_{\in X} = -y \Rightarrow y \in X \cap Y$$

$$(w+y)+x=0 \quad w+y \in X \cap Y \quad x \in \langle u_{k+1} \dots u_n \rangle$$

Seien w und y linear unabhängige Vektoren aus \mathbb{R}^n , d.h.

$$\text{ker } w = \{x \in \mathbb{R}^n \mid w \cdot x = 0\}$$

$$\text{ker } y = \{x \in \mathbb{R}^n \mid y \cdot x = 0\}$$

$$w + y = \{x \in \mathbb{R}^n \mid (w+y) \cdot x = 0\}$$

$$w + y = 0$$

$$\sum_{k=1}^m \alpha_k w_k + \sum_{k+1}^m \gamma_k y_k = 0$$

$$\sum \beta_j y_j = 0$$

$$\beta_j = 0 \quad \forall j$$

$$\downarrow \text{ind. d. } w_k \text{ u. } y_k$$

$$\alpha_i = 0 \quad \forall i$$

Lia

1) $B \in \langle A_1, \dots, A_n \rangle$

2) $B \neq 0$

B. $\exists j : \langle A_1, \dots, A_n \rangle = \langle A_1 \dots A_{j-1}, BA_{j+1} \dots A_n \rangle$

1) $\Rightarrow \langle A_1, \dots, A_n \rangle = \langle B, A_1, \dots, A_n \rangle$

1) $\Rightarrow B = \sum_i x_i A_i$ für offenbar x_i

2) \Rightarrow non triv; weil sonst x_i -posse annuliert
suppose the $x_1 \neq 0$

we 2 gru

$$B = \alpha_1 A_1 + \sum_2^n \alpha_i A_i$$

weiterreduziert
a A_2

$$A_1 = \boxed{\frac{1}{\alpha_1} B - \sum_2^n \frac{\alpha_i}{\alpha_1} A_i} \in \langle B, A_2, \dots, A_n \rangle$$

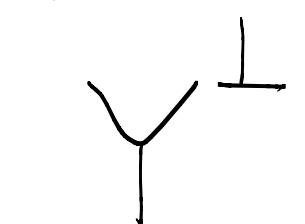
$\uparrow !!!$

Puis lemmme fondamental

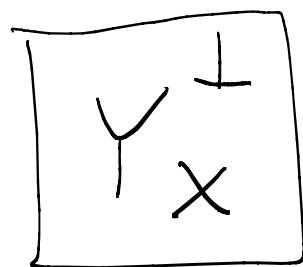
$$\underbrace{\langle A_1, \dots, A_n \rangle}_{\downarrow} = \langle B, A_1, \dots, A_n \rangle = \overline{\langle B, A_2, \dots, A_n \rangle}$$

X sp. retteneh

Y sottospazio di X

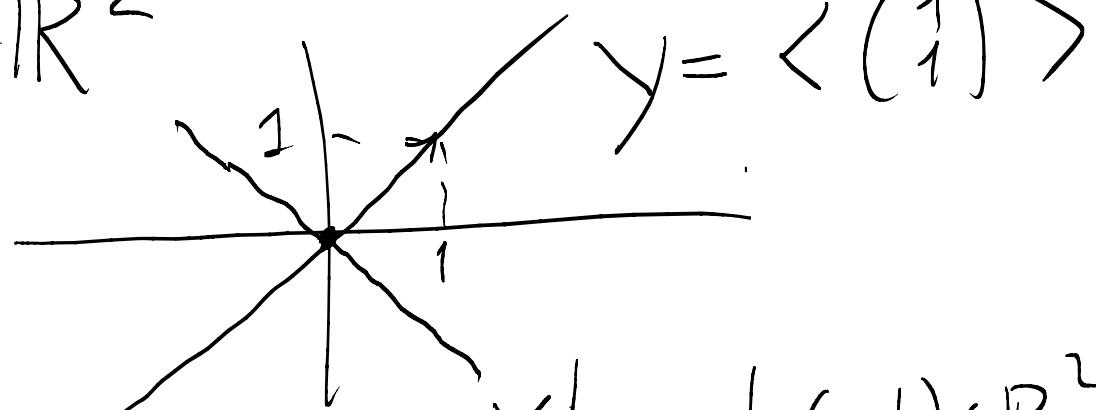


$$Y^\perp = \{x \in X : xy = 0 \ \forall y \in Y\}$$



l'insieme dei vettori di X ortogonali
a tutti i vettori di Y

$$X = \mathbb{R}^2$$



$$Y^\perp = \{(a, b) \in \mathbb{R}^2 : (a, b) \perp (1, 1)\}$$

$$a + b = 0$$

$$\boxed{a = -b}$$

$$Z = X \oplus Y$$

$$\boxed{1) Z = X + Y}$$

$$\boxed{2) \forall x \in X, \forall y \in Y \quad x+y=0 \Rightarrow x=y=0}$$

$$Y \text{ sottsp. } X \Rightarrow X = Y \oplus Y^\perp$$

Dim. 1) $X = Y + Y^\perp$ $x = x_Y + (x - x_Y)$
 $\perp Y$
 $\in Y^\perp$

$$y \in Y, y' \in Y^\perp : y+y'=0$$

$$|y|^2 = \underbrace{yy}_{\in Y} \underbrace{= 0}_{\in Y^\perp}$$

$$y, y' \in Y \cap Y^\perp \quad \boxed{y = -y'} \quad \text{multpl. by } y \\ |y| = y \cdot y = -y' \cdot y = 0 \Rightarrow \begin{array}{l} y = 0 \\ y' = 0 \end{array}$$

$$\dim X = \dim Y \oplus Y^\perp = \dim Y + \dim Y^\perp$$

u_1, \dots, u_n bilden $\dim X$

$$Y = \langle u_1, \dots, u_k \rangle$$

$$Z = \langle u_{k+1}, \dots, u_n \rangle$$

$$X = Y \oplus Z$$

Dnn:

$$1) X = Y + Z$$

jetzt $u_1, \dots, u_k, u_{k+1}, \dots, u_n$ bilden
generiert $\dim X$
(feste base)

$$2) y + z = 0, y \in Y, z \in Z \quad y \in Y \quad y = \sum_{i=1}^k \alpha_i u_i$$

$$z = \sum_{j=k+1}^n \beta_j u_j$$

$$0 = y + z = \sum_{i=1}^k x_i u_i + \sum_{j=k+1}^n \beta_j u_j$$

↓ y ↓ z
 $x_i = 0 \forall i$ $\beta_j = 0 \forall j$

$$y = \sum_{i=1}^k x_i u_i = 0$$

$$z = \sum_{j=k+1}^n \beta_j u_j = 0$$

für l^1 und l^2 Basis
 d. Rett. gl. u_i
 (beschreibt g^* von x)

Grenzen:

$$(w+ty) + x = 0$$

$$\in \underbrace{\langle w_1 \dots w_k \rangle}_{\in} \quad \underbrace{\langle x_{k+1} \dots x_n \rangle}$$

$$\Rightarrow \underbrace{x=0}_{w+y=0}$$

$$\langle w_1 \dots w_k \rangle \oplus \langle x_{k+1} \dots x_n \rangle = X$$

Sia u_1, \dots, u_n base di X . Siano $v_1, \dots, v_m \in X$ $m > n$

Ts. v_1, \dots, v_m sono dipendenti

DIM. $X = \langle u_1, \dots, u_n \rangle$

$$v_1 \in X \Rightarrow \exists \alpha_1, \dots, \alpha_n : v_1 = \sum_1^n \alpha_i u_i$$

Se $v_1 = 0$ il sistema v_1, \dots, v_m è dipendente. (Ts negata!)

Se $v_1 \neq 0$, per il lemma d' scambio

$$X = \langle u_1, \dots, u_n \rangle = \langle v_1, u_2, \dots, u_n \rangle$$

$$v_2 \in X = \langle v_1, u_2, \dots, u_n \rangle \quad v_2 = \beta v_1 + \sum_2^n \alpha_i u_i$$

Se fanno $\alpha_i = 0 \quad i=2..n \Rightarrow v_2 = \beta v_1$ e quindi v_1, \dots, v_m è dip.

Supponendo che
l'ipotesi delle
scambi sia
1.

Suppongo $\alpha_2 \neq 0$. Dal lemma di scambio

$$X = \langle v_1, u_2, \dots, u_n \rangle = \langle v_1, v_2, u_3, \dots, u_n \rangle$$

$$\overline{X} = \underbrace{\langle v_1 \dots v_k, u_{k+1}, \dots, u_n \rangle}_{\leftarrow} \Rightarrow v_{k+1} = \sum_1^k \alpha_i v_i + \sum_{k+1}^n \beta_j u_j$$

Se fanno $\beta_j = 0 \forall j = k+1 \dots n \Rightarrow \underline{v_{k+1}} = \sum_1^k \alpha_i v_i \Rightarrow v_1 - v_m \text{ d.t.}$

Altamente, se $\beta_{k+1} \neq 0$ e, per il lemma di scambio

$$X = \langle v_1 - v_k, u_{k+1} - u_n \rangle = \langle v_1 \dots v_k, v_{k+1}, u_{k+2}, \dots, u_n \rangle$$

Dopo n passi, si ottiene $X = \langle v_1 \dots v_n \rangle$ e perciò
 $v_j \in X \Rightarrow v_j \in \langle v_1 \dots v_n \rangle \quad \forall j = n+1, \dots, m$ il che è d.t.

AL_3.1

bisettore

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

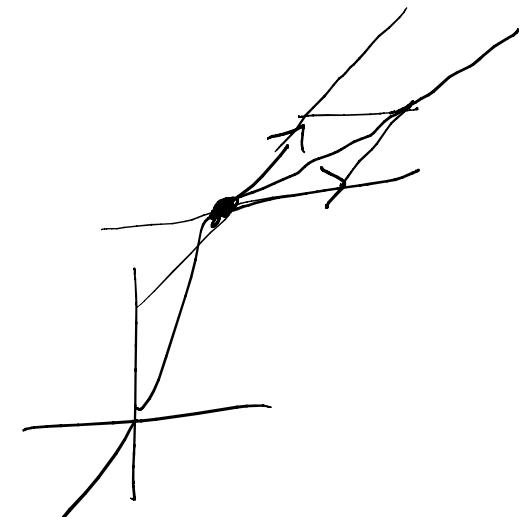
x_0

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

u

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

v



$$|u| = \sqrt{4} = 2$$

$$\hat{u} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} \quad \hat{v} = \hat{v}$$

$$\hat{u} + \hat{v} = \begin{pmatrix} 3/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$$

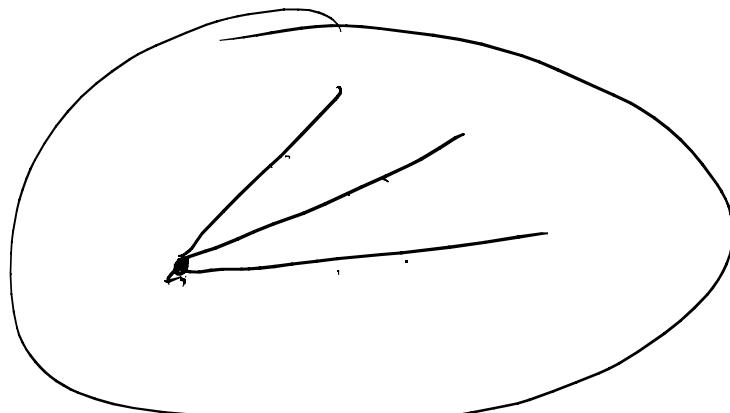
$$x(t) = x_0 + t(\hat{u} + \hat{v})$$
$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 3/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

↓ ↑

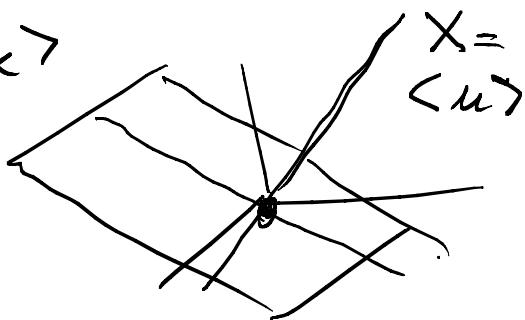
$t \in \mathbb{R}$

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$



$$X \subseteq \mathbb{R}^n$$

$$\langle u_1, \dots, u_k \rangle$$



$$x = \begin{cases} u_1 x = 0 \\ u_2 x = 0 \\ \vdots \\ u_k x = 0 \end{cases} \iff x \in X^\perp$$

$$ux = 0$$

$$u_1 x_1 + u_2 x_2 + u_3 x_3 = 0$$

$$\left[\underbrace{\sum_{i=1}^k \alpha_i w_i}_w + \underbrace{\sum_{j=k+1}^n \beta_j u_j}_x + \underbrace{\sum_{h=k+1}^m \gamma_h v_h}_y = 0 \right] \Rightarrow \begin{cases} \alpha_i = 0 \\ \beta_j = 0 \\ \gamma_h = 0 \end{cases} \text{? } i, j, h?$$

$$w \in X \cap Y \quad x \in \langle u_{k+1}, \dots, u_n \rangle \subseteq X$$

$$y \in \langle v_{k+1}, \dots, v_m \rangle \subseteq Y$$

$$\rightarrow w + x + y = 0 \Leftrightarrow \underbrace{w+x}_{\in X} = -y \in Y \Rightarrow y \in X \cap Y$$

$$\begin{aligned} & (w+y) + x = 0 \\ & \in X \cap Y \quad \in \langle u_{k+1}, \dots, u_n \rangle \\ & \quad " \\ & \quad \langle w_1, \dots, w_k \rangle \end{aligned}$$

AL-3.5

$$\beta_j = 0 \stackrel{\text{ind. } d \cdot u_{k+1} \dots u_n}{\Leftrightarrow} \sum \beta_j u_j = 0 \Leftrightarrow \boxed{x=0}$$

Th. 2. p. bese

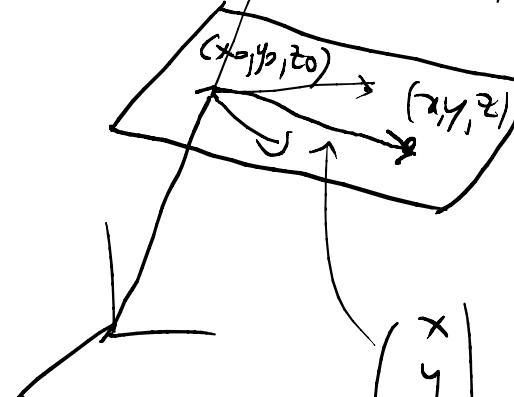
$$X = \langle w_1, \dots, w_k \rangle \oplus \langle u_{k+1}, \dots, u_n \rangle$$

$$\begin{array}{c} w+y \\ + \\ \Downarrow \\ w+y=0 \\ \Updownarrow \\ \alpha_i, \gamma_h = 0 \end{array}$$

$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ vettore normale $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ è il punto di percorso

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$\nu = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$



$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \underline{\underline{\begin{pmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{pmatrix}}}$$

1) Proiezione di x su $u \neq 0$

$$x_u = \frac{xu}{\|u\|^2} u$$

2) Proiezione x su $\langle u_1 \dots u_n \rangle$

$$x_{\langle u_1 \dots u_n \rangle} = \sum_i x_{u_i} u_i$$

$x_{u_i} = \begin{cases} 0 & i \neq j \\ \frac{xu_i}{\|u_i\|^2} & i = j \end{cases}$ se $i \neq j$
non negativo

$$x_{\langle u_1 \dots u_n \rangle} = \sum_i x_{u_i} u_i = \sum_i \frac{xu_i}{\|u_i\|^2} u_i$$

3) Proiezione su $\langle u_1 \dots u_n \rangle$ arbitraria.

$$x_{\langle u_1 \dots u_n \rangle} = \sum_i x_i u_i \text{ deve verificarsi per ciascuna}$$

\rightarrow
$$\left[x - \sum_i x_i u_i \right] u_j = 0 \quad \forall j = 1 \dots n$$

proiezione

Il vettore proiezione $\sum_i x_i u_i$ NON DIPENDE DALLA SOLUZIONE
 $x_1 \dots x_n$ scelta per calcolarlo

$$W = X \oplus Y \oplus Z$$

1) $W = X + Y + Z$

2) $x \in X, y \in Y, z \in Z \quad x+y+z=0 \Rightarrow x=y=z=0$

CNS

$\forall w \in W \exists_{w \in C} x, y, z : x+y+z=w$
je addende soms versch
 $w = x' + y' + z'$ $x=x' \quad y=y' \quad z=z'$

